DEPARTMENT OF MATHEMATICS

May 16, 2016

Report on the thesis of **Simeng WANG**

The general theme of the thesis is to develop Harmonic Analysis, more precisely the analysis related to Fourier series, in the setting of finite and compact quantum groups.

This thesis is composed of two parts: the first part is devoted to L_p-improving convolution operators on compact quantum groups (the main analysis being actually on the finite, i.e. finite dimensional ones) and the second one on Fourier analysis of multipliers and the so-called thin sets on compact quantum groups

Quantum groups provide, as is well known (since Kac, Woronowicz, and so on), a more satisfactory picture of the duality of locally compact non-Abelian groups. Indeed, for instance the dual of a finite group is not in general a group, but the notion of finite quantum groups allows one to correct this by extending what is meant by "a group".

This is illustrated very well by the first part in the thesis, devoted to L_p-improving operators. Previous results due to Ritter treated separately the case of multipliers and convolutors on finite groups, but Wang manages in his thesis to give a unified treatement for finite quantum groups from which he can deduce both results of Ritter just from a single more general statement. This is a very satisfactory point of view and a very nice result. Moreover, by a careful analysis of the reduced free product of two finite quantum groups, he is able to produce L_p-improving operators also on such free products, which of course are now infinite dimensional.

Let me comment more on the second and main part of the thesis. There Wang introduces and studies various sorts of interpolation sets for multipliers on L_p for $1 \leq p \leq \infty$, and he is naturally led also to notions of Sidon sets (for $p \in \{1, \infty\}$) and $\Lambda(p)$-sets (for $p \in (1, \infty)$) in analogy with Rudin's classical definition. As in the classical case of compact groups the "thin sets" are subsets of the set of irreducible representations on a compact quantum group $\mathbb{G}$.

The basic example of a compact quantum group is $SU_q(2)$ $(0 < q < 1)$. This is a

deformation of the classical commutative C^*-algebra $C(G)$ formed of all continuous functions on $G = SU(2)$, but the deformation is a non-commutative C^*-algebra equipped with a state analogous to Haar measure but that is not tracial. This is a typical example. More generally for any simply connected semi-simple compact Lie group one can define a "compact quantum group" G_q. While in some sense one can define an analogue of Fourier analysis (Peter-Weyl style) on G_q that resembles the well known one of G, the questions that are addressed in this thesis require quite sophisticated techniques from the so-called modular theory (originating with Tomita-Takesaki). In particular the space $L_p(G_q)$ (or more generally $L_p(\mathbb{G})$ for any compact quantum group $\mathbb{G}$) is defined using the latter theory. In general the fact that the Haar state is not tracial creates specific technical problems and leads to interesting considerations by the author. Only when the group is "of Kac type " the Haar state is tracial, but nevertheless some sets of representations can be such that the Haar state roughly behaves like a tracial state on the linear span of its entries. In fact the author shows in the Prop. 4.5 of the thesis that it is the case whenever the set is an interpolation set of any of the kinds he is considering. This leads him in Prop. 4.17 to the proof that there are no infinite Sidon or $\Lambda(p)$-set for $1 < p < \infty$ for $SU_q(2)$, just like for the classical $SU(2)$. Nevertheless, in Prop. 4.16 he produces an example of central $\Lambda(4)$-set for $SU_q(2)$.
This is of interest because it is in sharp contrast with the observation that there are no infinite central Sidon sets in $SU_q(2)$ or in $\mathbb{G}_q$. The reason is simply that the fusion rules for the irreps being the same ones as for $SU(2)$ or G (i.e. when $q = 1$), the central Sidon sets must be the same independently of $0 < q \leq 1$. So the statement follows simply from the known fact that $SU(2)$ or G do not admit infinite central Sidon sets (in their dual objects).

In a very nice final appendix the author, generalizing results that are well known in the tracial framework, shows that arbitrary bounded orthonormal systems contain infinite sequences that are of $\Lambda(p)$ type for all $1 < p < \infty$. This requires a quite skillful estimate of the different terms that appear when one develops the L_p-norm in a general non-tracial Haagerup L_p-space, for p an even integer. As the author notes the same ideas can be applied to the Kosaki version of non-commutative L_p-space. This result implies the (not too surprising) fact that if $\mathbb{G}$ has an infinite subset with d_π uniformly bounded (this is the analogue of irreps with uniformly bounded dimensions) then, for any $1 < p < \infty$, there is a further subset that is a $\Lambda(p)$-set.

In §4.1, the author has obtained in the quantum setting the various characterizations of Sidon sets as interpolation sets. He proves the (not too surprising) fact that only in the finite dimensional case can the whole dual quantum group $\hat{\mathbb{G}}$ be Sidon. He observes that the union of two Sidon sets that are mutually free in a reduced free product remains Sidon. A similar result holds for arbitrary unions of mutually independent Sidon sets, i.e. sets in $\hat{\mathbb{G}}$ when $\mathbb{G}$ is the minimal tensor product of a family $(\mathbb{G}_i)_{i\in I}$ of compact quantum groups.

Overall, the thesis is very well written, very detailed, the results are valuable contributions, very carefully presented and quite interesting.

His results very significantly improve previous results notably those by Blendek and Michaliček (J. Oper. Th. 2013), that were restricted to the Kac type.

The author shows that he masters a broad spectrum of very delicate techniques, and it is clear when one reads the details that all is very much fine tuned and precise. It is clearly the thesis of someone who has a very solid understanding of every aspect of the subject. I find the command with which the author masters all the technical aspects of this difficult subject quite impressive.

One can only regret that certain questions are not discussed, e.g. the union problem for Sidon sets and related questions. But these are probably not too accessible, given that even for classical compact groups, although the result was known, there was no available published proof until recently. Actually, in any case, it is not clear at all how to tackle the quantum case, but perhaps at least the problem should have been raised.

Another criticism is the lack of examples, but again this criticism is valid also for the classical theory of thin sets in dual objects of compact non Albelian groups.

My global opinion is completely positive. This is a very good thesis. I recommend to accept it for the requirement of "Thèse de Doctorat" at Université de Franche-Comté and also as a PhD thesis at University of Warsaw.

Gilles Pisier
Distinguished Professor
(A.G. and M.E. Owen Chair)

3368 TAMU
College Station, TX 77843-3368

Tel. 979.845.3261 Fax. 979.845.6028
www.math.tamu.edu/mathdept@tamu.edu