Report on PhD Thesis of MSc Tomasz Odrzygóźdź

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The following is my report on the thesis of Tomasz Odrzygóźdź:

Bent walls in random groups

When studying any mathematical object, in addition to asking what properties are *possible* it is also natural to ask what properties are *typical*. In the case of groups, such questions have been a field of rapid development and interest over the past four decades, with particular momentum given by Gromov's introduction in 1993 of the "density model" for random groups. This model, in a fashion analogous to the idea of a random graph, considers groups defined by presentations chosen at random from all presentations with m generators and n relations of length k, where $m \geq 2$ is fixed, k is a parameter that goes to infinity, and $n = (2m-1)^{kd}$ grows exponentially with k at a rate determined by a fixed 'density' $d \in (0,1)$. If a property holds with probability going to 1 as $k \to \infty$ one says that the property holds "asymptotically almost surely" (a.a.s.).

A source of great interest in this model is the fact that different properties appear at different densities: for example such groups are (a.a.s.) trivial or C_2 at $d > 1/2$ while they are infinite hyperbolic groups at densities $d < 1/2$ [Gromov 1993, Ollivier 2004]. Another dichotomy appears when considering Kazhdan's property (T): at densities $d > 1/3$ a.a.s. there is property (T) [Żuk 2003, Kotowski-Kotowski 2013], while at densities $d < 5/24$ there is no property (T) [Ollivier-Wise 2011, Mackay-Przytycki 2015]. Exactly when property (T) appears remains an important open question, which leads to the topic of Odrzygóźdź's thesis.

The way in which property (T) is shown for the Gromov density model is via a different random group model studied by Żuk, called the "triangular density model": here relations are all of length $k = 3$, and one lets the number of generators m go to infinity (with $n = (2m-1)^{kd}$ as before). One can show that, roughly, groups in the Gromov density model are quotients of groups in the triangular model at the same density.

By work of Żuk [2003] and Antoniuk-Łuczak-Świątkowski [2015] at densities $d > 1/3$ random triangular groups have property (T) (and with work the result for the Gromov model follows), while at $d < 1/3$ they are free, and so have a strong negation of property (T), the Haagerup or a-T-menable property: they have a proper affine isometric action on Hilbert space.

Given the success of the triangular density model in shedding light on random groups, it is an interesting and important question to consider what can be understood by using random relations of other fixed lengths, hence the "k-gonal model" considered by Odrzygóźdź.

The first main result of the thesis, Theorem A, is the existence of a sharp threshold in the hexagonal model ($k = 6$), at densities $d > 1/3$ the group has property (T) while at densities $d < 1/3$ is does not (a.a.s.). A general strategy to show that a group G does not have property (T) is to find a non-trivial action of G on a CAT(0) cube complex [Niblo-Roller 1998]. Such actions can be found using Sageev's construction from an action of G on a space with walls. The prototypical construction of such walls in this area is found in Wise's proof of cubulation for small cancellation groups [2004]: join antipodal edges of faces in the Cayley complex of the group. The problem with this strategy at higher densities is that the resulting 'walls' are not embedded. Odrzygóźdź's solution is to 'bend' the walls at collision points by modifying their construction. Up to density $d < 1/3$ he shows the resulting walls are embedded trees, giving the desired result. (That these groups have property (T) for $d > 1/3$ follows fairly quickly by considering the hexagonal model groups as, roughly, quotients of triangular groups.)

This result is interesting, original and technically challenging. I fully expect it to be publishable in a leading international journal. While the idea of modifying walls in a different model had appeared before [Mackay-Przytycki 2015], the methods required to study this model are significantly different. Odrzygóźdź has to set up a sophisticated machine to deal with walls with self-intersections, using new variations of Ollivier's linear isoperimetric inequality for random groups, and careful arguments with not-necessarily-planar diagrams.

The techniques developed also apply to the square model ($k = 4$), giving the new result that property (T) fails for $d < 3/8$, Theorem B. This strengthens the author's earlier bound of $d < 1/3$ [2014], which was achievable with Ollivier-Wise walls, but could not be improved without his new 'bent' wall approach. Again, this is state-of-the-art for this model. It is the first time in this family of models that we can see behaviour 'non-monotonic' in k: when $1/3 < d < 3/8$ in the case $k = 3$ or 6 we have (T) but for $k = 4$ we do not.

The final main result of the thesis, Theorem C, shows that one can go further than just show failure of property (T) in the square model, and show the strong negation of property (T), the Haagerup property. This follows by finding walls so that the resulting action on the cube complex is proper and cocompact [Chatterji-Niblo 2005]; this means that any two points in the space must be separated by walls, where the number of walls grows uniformly with the distance between the points.

Odrzygóźdź shows that in the square model up to density $d < 3/10$ groups act properly and cocompactly on a CAT(0) cube complex. The method here is to again use bent walls, and to show that along a geodesic connecting points enough of the transverse walls will be distinct. This is a major result since, by Agol's work this implies that such groups are virtually special, and so linear and residually finite, as well as Haagerup. The paper containing Theorem C and Corollary D has now appeared in the leading journal "Israel Journal of Mathematics".

(In other developments, after Odrzygóźdź's proof of Theorem C and Corollary D initially appeared, Yen Duong independently achieved the same results up to densities $d < 1/3$ using as key ingredients his 2014 work and a new breakthrough by Groves and Manning extending Agol's work.)

Throughout the thesis arguments are made carefully and thoroughly, with helpful figures and motivation for the techniques used. There are many original ideas required to deal with technical problems. As is inevitable with these technical arguments, there are some minor points for clarification or correction which I have attached for the author to consider, but overall the quality of the writing is good.

This is an excellent thesis, contributing new understanding, tools and insight to problems of international interest, and certainly deserving of a PhD.

Minor comments and corrections

The following are minor comments and corrections.

Page 4, Problem 1.2.3 I don't think the existence of a sharp threshold (in your sense) is known in the Gromov model? There's some threshold function by general random structures results, but I don't know why it must be of the form $f(l) = (2m-1)^{dl(1+o(1))}$.

Page 5, Theorem 1.2.4 You have geometric dimension at most 2.

Page 8, Line 24 "any subcomplex of a diagram with K-small hull is also a diagram with K-small hull" — not clear what exactly you mean here

Page 9, (2.6) "Cancel$(Y) \leq \sum_{f\in Y^{(2)}} \delta(f)$..." to cover the case some edges have degree 0.

Page 11 Proof of Lemma 2.1.2: "there are finitely many k-gonal complexes with at most B faces" — what if there are edges with degree 0?

Line 19: $C(K,m)$ should be $C(B,m)$

Page 12 Line 13: One '∂Z' should be '$|\partial Z|$'

Line -5: "..., so $kA \leq |\partial Z| \leq k|Y|$, thus..."

Page 14, Line 4 Add something like "Later we use the following lemma."

Page 15 'cocylces'

Page 17 Definition 3.2.4: Clarify if you allow multiple edges between x and y if there are multiple 2-cells?

Definition 3.2.5: For $1 \leq i < n$, x_i and x_{i+1} are antipodal midpoints. And $\Lambda_{int} = \lambda_{int}$?

Page 21 Lines 17, -4: c_A is c'?

Page 24, line 3 "$G^+ =$"

Page 25, line 13 'where' is 'were'

Page 34 Line 2: 'Bu' is 'By'

Proof of 4.2.2.: Some confusion between 'D' and '$\mathcal{D}$'

Page 35 Line 13: "...and A''. Since"

Line 15: Worth clarifying where $2|\gamma| + 6(2|\gamma|+2)$ comes from.

Page 38, line -2,-1 "not more than $\frac{1}{2(1-3\delta-\epsilon)}$ 2-cells, for any fixed $\epsilon > 0$, w.o.p.." or similar.

The same change on Page 46, line -4.

Page 40, line -15 "exactly one edge of three distinct standard hypergraphs"?

Page 41, Remark 4.2.10 Why if they are connected by a standard hypergraph must they be connected by a bent hypergraph?

Page 43, line -9 'end' should be 'and'

Page 44, line 20 $m := (2n-2)$ safer

Page 45, line 1 I suppose also without loss of generality the neutral element is the first vertex of e?

Page 46, line -6 Could clarify "considering a boundary deviation of $|\tilde{\partial}\mathcal{D}_{\mathcal{T}}| - |\mathcal{D}|$ instead"

Page 48, line 4 "join $x, y \in V$ by an edge"

Page 52, line -9 "of $\{E_1, E_2, \ldots, E_l\}$ in blue."

Page 54, line -1 "all distinguished 2-cells $\mathfrak{c}$ in $\partial_2(\mathcal{D})$"

Page 55 Line 3 "for each $\mathfrak{c} \in \partial_2^s(\mathcal{D})$ we glue"

Line 19: "projection map $p : Y' \cup Y'' \to \mathcal{D}^h$" ?

Line 22: "Since horns have disjoint edges" – explain why

Line 23: "$4(|\gamma| + 1)$"?

Page 57 Line 24: "$|\mathcal{A}| = 2$"

Line -12: No "Proposition Proposition"

Page 58 Line 2: "adding horns" – clarify you add them to the carrier only? Or also 2-cells adjacent to γ if that's what you want.

Line -10: "will be called a"

Page 61 Line -10: "are dual to"

Lines -6 to -4: $|\partial\mathcal{E}|$ should be $|\tilde{\partial}\mathcal{E}|$

Page 63 Are Λ' and Λ equal?

Page 66 Lines -8,-2: Is it possible that $\mathfrak{c}$, although in the carrier of Λ_y, is not in the carrier of Λ_{sy}?

Lines -4,-2: Λ^x, Λ^y are Λ_x, Λ_y

Page 67, line 4 "according to Proposition 5.1.1. and Theorem 2.0.6"?