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Report on the doctoral thesis by Tomasz Pełka « Smooth $\mathbb{Q}$-homology planes satisfying the Negativity Conjecture »

On the subject

The thesis of Tomasz Pełka is devoted to the following classification problems :
— classification of smooth $\mathbb{Q}$-acyclic algebraic surfaces, and
— classification of rational cuspidal curves in the projective plane.
The varieties in the thesis are defined over the complex number field $\mathbb{C}$. The former classification problem is discussed in Chapters 2–4, and the latter one in the last Chapter 5. The both subjects are ultimately related. Indeed, the complement to a rational cuspidal curve in $\mathbb{P}^2$ is a smooth $\mathbb{Q}$-acyclic algebraic surface.

A manifold is called $\mathbb{Q}$-acyclic if its homology groups are torsion groups. The interest to the subject was initiated by the pioneer paper by Ramanujam [Ram71] [1]. In his attempt to construct a counterexample to the Zariski cancellation problem in the two-dimensional

1. We refer here to the list of literature in the thesis.

case, Ramanujam found a surprising example of a smooth complex affine surface which is topologically contractible, but not simply connected at infinity, and so, is not isomorphic (or even homeomorphic) to the complex affine plane $\mathbb{C}^2$. Later on, in the work by Miyanishi-Sugie and Fujita (1979-1980), the Zariski cancellation problem in the two-dimensional case was resolved in the positive sense. Many more surfaces of Ramanujam type were constructed by tom Dieck [tD90, tD92], tom Dieck and Petrie [tDP89, tDP91, tDP93], Sugie [Sug90], in the thesis by Mara Neusel [Neu92], by the reviewer[2],[3],[4],[5], e.a. In the latter papers, these surfaces served in order to construct an infinite countable collection of deformation families of exotic complex algebraic structures on $\mathbb{R}^{2n}$, $n \geq 3$. By an *exotic complex algebraic structure* on $\mathbb{R}^{2n}$ one means a smooth, complex affine variety diffeomorphic to $\mathbb{R}^{2n}$ and non-isomorphic to $\mathbb{C}^n$.

Another spectacular appearance of topologically contractible (not necessarily smooth) complex affine surfaces is related to the classification problem for the $\mathbb{C}^*$-actions on $\mathbb{C}^n$. This problem was discussed, in particular, in a series of papers of Koras and Russell (1984–2008). These authors composed a certain list of exotic *Koras-Russell threefolds* X equipped with a (non-linearizable) $\mathbb{C}^*$-actions. The associated categorical factors occur to be topologically contractible affine surfaces. The exoticity of the Koras-Russell threefolds, proven by Kaliman and Makar-Limanov (1997), implies that any $\mathbb{C}^*$-action on the standard $\mathbb{C}^3$ is linearizable. Notice that the linearization problem for $\mathbb{C}^*$-actions on $\mathbb{C}^n$ remains open for $n \geq 4$.

There are different approaches to the classification of smooth $\mathbb{Z}$- and $\mathbb{Q}$-acyclic surfaces. The one initiated by tom Dieck and Petrie [tDP89] deals with an algorithmic construction of such surfaces starting with certain special configurations of rational curves in the complex projective plane. The $\mathbb{Q}$-acyclic surface which can be obtained starting with a line configuration were completely classified by tom Dieck and Petrie [tDP89]. However, they do not exhaust all the posible $\mathbb{Q}$-acyclic surfaces. Tom Dieck [tD92] conjectured that every such surface arises from a configuration of lines and conics in $\mathbb{P}^2$. In her thesis [Neu92], Lara Neusel elaborated an (eventually, non-complete) list of lines-conics configurations which lead to $\mathbb{Q}$-acyclic surfaces. *One of the main results of the thesis by Tomasz Pełka confirms the tom Dieck conjecture under an additional assumption that Palka's Negativity Conjecture holds.*

2. M. Zaidenberg, *Affine acyclic curves and surfaces. Exotic algebraic structures on* $\mathbb{C}^n$, 29.Arbeitstagung, Bonn, 23.–29.Juni 1990. Max–Planck–Institut für Mathematik, Bonn, 1990. Preprint MPI/90–52, 5pp.

3. M. G. Zaidenberg, *Ramanujam surfaces and exotic algebraic structures on* $\mathbb{C}^n$, $n \geq 3$, Soviet Math. Dokl. 42 (1991), 636–640.

4. M. G. Zaidenberg, *On Ramanujam surfaces,* $\mathbb{C}^{**}$*–families and exotic algebraic structures on* $\mathbb{C}^n$, $n \geq 3$, Trans. Moscow Math. Soc. 55 (1994), 1–56.

5. M. Zaidenberg, *An analytic cancellation theorem and exotic algebraic structures on* $\mathbb{C}^n$, $n \geq 3$, Astérisque 217 (1993), 251–282.

The second approach consists in describing the discrete invariants of a $\mathbb{Q}$-acyclic surface, such as the weighted dual tree of the boundary divisor, and the continuous parameters via the deformation theory, see, e.g., Flenner and the reviewer [FZ94], the monograph [Miy01] and the references therein; see also Palka [Pal11a–Pal13] for a classification of singular $\mathbb{Q}$-acyclic surfaces and [Pal11b] for a survey.

In parallel, one could try to classify as well the topologically contractible curves on the $\mathbb{Z}$- and $\mathbb{Q}$-acyclic surfaces. The latter was done :

— for $\mathbb{C}^2$ in the work of Abhyankar–Moh and Suzuki [Suz74, AM75], V. Lin and the reviewer [LZ83];
— for $\mathbb{Z}$-acyclic surfaces by the reviewer [6];
— for $\mathbb{Q}$-acyclic surfaces by Miyanishi–Sugie [MS91], Miyanishi–Tsunoda [MT92], and Gurjar–Miyanishi [GM92]; see the monograph [Miy01] for a systematic exposition.

It occurs that, modulo the automorphism group action, a $\mathbb{Q}$-acyclic surface can contain just few such curves.

A rational cuspidal curve C in $\mathbb{P}^2$ is a rational plane curve with only unibranched singularities. As it was mentioned already, for such a curve C the affine surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic, i.e., $H_i(\mathbb{P}^2 \setminus C, \mathbb{Q}) = 0$ for $i > 0$. The classification of rational cuspidal plane curves up to the action of the automorphism group $\mathrm{PGL}(3, \mathbb{C})$ on $\mathbb{P}^2$ is an old and well established subject, and, at the same time, a difficult open problem with many interesting connections. It is sufficient to cite the papers by Fernandez de Bobadilla–Luengo–Melle-Hernandez–Nemethi [FdBL$^+$07], Bodnar [Bod16a-b], Borozdik–Livingston, Borodzik–Zoladek [BZ10], Cotterill et al., Dimca–Sticlaru, Fenske [Fen99a-b], Flenner–the reviewer [FZ96-00], Gurjar–Miyanishi [GM96], Koras–Palka [KP16], Moe [Moe08-14], Orevkov [Ore02], Palka [Pal19], Piontkowski [Pio07], Matsuoka–Sakai, tom Dieck–Petrie [tDP93], Tono [Ton00a-12c], C. T. C. Wall [Wal04], Yoshihara [Yos88], et al.

Several infinite countable series of such curves, as well as several sporadic examples were described in the literature, and some difficult conjectures were formulated. These are confirmed partially in computer stimulations and in some special cases. For instance,

— the Rigidity Conjecture of Flenner and Zaidenberg [FZ00] asserts that a rational cuspidal curve C with the complement $\mathbb{P}^2 \setminus C$ of log-general type is projectively rigid and has unobstructed deformations. This leads to certain numerical identities, which hold indeed in all known examples;
— the Coolidge–Nagata Conjecture suggests that any rational cuspidal plane curve can be sent to a line by means of a Cremona transformation. The conjecture was proven in a series of papers by Palka and Koras–Palka, see [KP17];

6. M. G. Zaidenberg, *Isotrivial families of curves on affine surfaces and characterization of the affine plane*, Math. USSR Izvestiya 30 (1988), 503–532. Additions and corrections, ibid., 38 :2 (1992), 435–437.

— the strongest possible conjecture suggests that, besides the known series, there is only a finite number of sporadic rational cuspidal plane curves with the complement of log-general type;
— the Orevkov–Piontkowski Conjecture [Pio07] says that a rational cuspidal plane curve can have at most 4 cusps, and the only such curve with 4 cusps is the 4-cuspidal quintic. Tono [Ton05] established the upper bound 8 for the number of cusps; Palka [Pal19] succeeded to decrease this bound to 6. In a recent preprint by Koras and Palka[7], a proof of the Orevkov–Piontkowski Conjecture has been announced.
— Palka's Negativity Conjecture asserts that for a rational cuspidal plane curve C, the Kodaira–Iitaka dimension of $K + 1/2D$ is negative, where D is the reduced total transform of C after the embedded resolution of its cusps.

One of the main results of the thesis is the complete classification of the rational cuspidal plane curves verifying the Negativity Conjecture.

On the content

A *smooth log-surface* is a pair (X, D), where X is a smooth projective surface, and D is a reduced divisor on X with simple normal crossings (an SNC-divisor). Any smooth $\mathbb{Q}$-acyclic surface S is isomorphic to the complement $X \setminus D$ for a rational smooth log-surface (X, D) (see [GP99]), where all the irreducible components of D are smooth rational curves, and the dual graph of D is a tree. For instance, if $S = \mathbb{P}^2 \setminus C$, where $C \subset \mathbb{P}^2$ is a rational cuspidal curve, then one can choose for (X, D) the minimal embedded resolution of the singularities of C. The adjoint divisor $K+D$, where $K = K_X$ is the canonical divisor, plays an essential role in the study of the affine surface $S = X \setminus D$. Its Kodaira–Iitaka dimension, that is, the log Kodaira dimension $\kappa(S)$ is a major numerical invariant of S. The Rigidity Conjecture predicts that $K(K + D) = 0$, and, moreover, the higher cohomology groups of the logarithmic tangent sheaf of (X, D) vanish provided S is of log-general type, that is, $\kappa(S) = 2$.

Notice that the classification of smooth $\mathbb{Q}$-acyclic surfaces not of log-general type is known (Miyanishi–Tsunoda, 1982). Their classification in the case $\kappa(S) < 0$ ($\kappa(S) = 1$, respectively) uses the result due to Iitaka and Kawamata (1978) saying that such a surface S admits a pencil of curves with the general member $\mathbb{A}^1 \cong \mathbb{C}$ ($\mathbb{A}^1_* \cong \mathbb{C} \setminus \{0\}$, respectively). The classification of smooth $\mathbb{Q}$-acyclic surfaces with $\kappa(S) = 0$ is due to Fujita [Fuj82]. Therefore, the $\mathbb{Q}$-acyclic surfaces considered in the thesis of Tomasz Pełka are assumed to be of log-general type.

7. M. Koras and K. Palka. *Complex planar curves homeomorphic to a line have at most four singular points.* arXiv :1905.11376.

The Koras–Palka's solution of the Coolidge–Nagata Conjecture explores systematically the adjoint $\mathbb{Q}$-divisor $K+1/2D$ on X. A natural extension of Palka's Conjecture says that $\kappa(K+1/2D)<0$ for any smooth log-surface (X,D) provided the complement $X\setminus D$ is $\mathbb{Q}$-acyclic. This conjecture is still open. *The objective of the thesis is to provide complete lists of rational cuspidal curves and of smooth $\mathbb{Q}$-acyclic surfaces assuming the (extended) Negativity Conjecture.*

The classical Castelnuovo-Enriques classification of smooth projective surfaces amounts to classify the minimal surfaces of non-maximal Kodaira dimension. The approach to the classification of $\mathbb{Q}$-acyclic surfaces in the thesis is also according to the log Kodaira dimension. The assumption $\kappa(K+D)<0$ is replaced here by a weaker one $\kappa(K+\frac{1}{2}D)<0$. Under the latter assumption, the case $\kappa(K+D)=2$ considered in the thesis is the most delicate; for $\kappa(K+D)<2$ the classification is known.

Let us start with the last Chapter 5 devoted to the classification of rational cuspidal plane curves. It is based on the joint papers by Palka and Pełka [PP17-18].

K. Palka [8] elaborated a new logarithmic version of the Mori Minimal Model Program (MMP), called the *almost MMP*, which is well adapted to the study of rational cuspidal plane curves. Under the Negativity Conjecture, the surface X admits a pencil of rational curves meeting D in $m(C)\leq 4$ points, where $m(C)$ is chosen to be the minimal possible. This pencil plays a distinguished role in the classification. The inequality $m(C)\leq 2$ means that the complement $\mathbb{P}^2\setminus C$ admits an $\mathbb{A}^1$- or $\mathbb{A}^1_*$-pencils, and so, is special, that is, not of log-general type. In the case $m(C)=3$ the complement $\mathbb{P}^2\setminus C$ admits a $\mathbb{C}^{**}$-pencil, and the almost MMP terminates by a contraction of fiber type, while in the case $m(C)=4$ it terminates with a log Del Pezzo surface.

The rational cuspidal plane curves whose complements admit $\mathbb{C}^{**}$-pencils are classified completely (Palka and Pełka [PP17]). It occurs that these curves satisfy the Negativity Conjecture. In [PP18], the log-Del Pezzo case is studied, and a criterion for such a curve to satisfy the Negativity Conjecture is established. As an outcome, the complete list of all the rational cuspidal plane curves which satisfy the Negativity Conjecture is elaborated. This classification is summarized in Theorem 1.6 of the thesis. Besides the previously known examples, the list includes a new sequence of bicuspidal curves with unbounded, along the sequence, difference « the degree minus the highest multiplicity of the cusps ». All the curves in the list occurred to be projectively rigid, hence verify the Rigidity Conjecture. The maximal number of cusps of such a curve is 4, and the four-cuspidal quintic is a unique curve in the list with 4 cusps. Thus, an important corollary of the classification in Chapter 5 says that both the Rigidity and the Orevkov–Piontkowski Conjectures hold

8. K. Palka, *The Coolidge–Nagata conjecture, part I*, Adv. Math. 267 (2014), 1–43.

modulo the Negativity Conjecture, see Corollaries 1.4 and 1.7. Besides, for any cuspidal curve C from the list, all the plane curves E with $E \setminus C \cong \mathbb{A}^1_*$ are described. It is established that there exist at least one and at most two such curves E in $\mathbb{P}^2$. The proofs are rather technical and involve a number of case-by-case computations. This extends the known results on classification of the topologically contractible curves on $\mathbb{Z}$- and $\mathbb{Q}$-acyclic surfaces (notice that $\mathbb{A}^1_*$ is not topologically contractible).

The same line of arguments works also in a more general setup of smooth $\mathbb{Q}$-acyclic surfaces satisfying the Negativity Conjecture. Their classification is summarized in Theorem 1.2; the proofs are done in Chapters 2 and 3 of the thesis. Chapter 2 is devoted to a thorough description of the almost MMP for a smooth $\mathbb{Q}$-acyclic surface satisfying the Negativity Conjecture. The author adopts the Palka's algorithm of the almost MMP, but modifies it in his more general setting, see Remark 2.20 in the thesis. This brings important advantages. The major advantage of the modified algorithm is the fact that the final objects can be classified. However, the guiding principle remains the same. Indeed, there is a dichotomy according to the outcome of the almost MMP, see Proposition 2.25. Namely, either the initial smooth $\mathbb{Q}$-acyclic surface carries a $\mathbb{C}^{**}$-pencil, or otherwise the outcome of the modified almost MMP is a singular log Del Pezzo surface of Picard rank one with only A_n-singularities, hence Gorenstein. The smooth $\mathbb{Q}$-acyclic surfaces with a $\mathbb{C}^{**}$-pencil where classified by Miyanishi and Sugie [MS91]. A classification of (prime) singular log del Pezzo surfaces is still unknown. However, there exists a classification of Gorenstein log del Pezzo surfaces elaborated by Furushima [Fur86]. The paper [Fur86] contains certain inaccuracies, discovered and corrected by Tomasz Pełka, see the footnote on p. 17 of the thesis. In fact, Tomasz Pełka shows that one can restrict to just three weighted projective planes from the Furushima's list, namely, $\mathbb{P}^2$, the quadric cone $\mathbb{P}(1,1,2)$ (replaced by the Hirzebruch surface $\mathbb{F}_2$), and $\mathbb{P}(1,2,3)$, see Lemma 3.2.

This provides a way to grow a given prime Gorenstein log del Pezzo surface starting with a certain configuration of rational curves in $\mathbb{P}^2$, as it suggests the tom Dieck-Petrie program. In fact, a finite, short list of configurations of lines and conics with the sum of degrees at most 11 is enough. Chapter 3 of the thesis provides a detailed examination of all possible initial arrangements of plane rational curves under the Negativity Conjecture. This provides a new effective bound in the Finiteness Conjecture of the reviewer for the considered class of $\mathbb{Q}$-acyclic surfaces. In Chapter 4 the consequences of the classification for the classical conjectures are derived, see Corollary 1.4. Besides, the automorphism groups of the smooth $\mathbb{Q}$-acyclic surfaces are listed, see Corollary 1.5.

It is worthwhile to stress that the passage between a minimal compactification of a given smooth $\mathbb{Q}$-acyclic surface and a configuration of plane rational curves is a cumbersome business. It is illustrated on several examples in Section 2.7. A brilliant simplifying

observation says that, in order to describe the initial curves arrangements in $\mathbb{P}^2$, it suffices to stick to a special class of the smooth $\mathbb{Q}$-acyclic surfaces such that all the (-1) curves in a special position with respect to the boundary divisor, which appear during the almost log MMP contraction procedure, must be contracted; see Lemma 3.1.

The author observes that there is an objective reason to distinguish between the two classification problems, for $\mathbb{Q}$-acyclic surfaces and for rational cuspidal curves. Indeed, upon the classification of rational cuspidal curves, the proper transform of the given curve in the embedded resolution of its singularities should not be contracted under the almost MMP. In the case of a general $\mathbb{Q}$-acyclic surface, there is no such a special component of the boundary divisor. This creates an additional ambiguity in the choice of an almost MMP, and makes the process more complicated.

The thesis on somewhat less than 90 pages is a result of a hard work, both theoretical and computational. The expositions, especially the proofs, are sometimes rather compressed and laconic (while correct). An extremal example of this is, for instance, the proof of Lemma 3.2, which needs much effort from the reader to reconstruct the details.

Evaluation

The thesis of Tomasz Pełka is devoted to the classification problems for $\mathbb{Q}$-acyclic surfaces and plane rational cuspidal curves. The author succeeded to accomplish the classification under the additional Negativity Conjecture, which holds indeed in all known examples of such objects. As a consequence, he proved that several famous, difficult conjectures (the Strong Rigidity Conjecture, the Finiteness Conjecture, the Orevkov–Piontkowski and tom Dieck-Petrie Conjectures) hold under the Negativity Conjecture, and obtained several nice, unexpected new results. All these results constitute a formidable breakthrough in an active area of mathematical research.

In my opinion, the doctoral thesis by Tomasz Pełka meets the highest standards. At present, Tomasz Pełka possesses a deep knowledge in the subject. He mastered a reach instrumentarium of algebraic geometry. In conclusion, Tomasz Pełka definitely merits to confer him the degree of doctor of mathematical sciences.

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