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Maps between classifying spaces of the unitary groups

Doctoral dissertation summary

The dissertation is devoted to a classical problem of Homotopy Theory, namely homotopy classification of maps between classifying spaces of compact Lie groups. The main result is a construction of new exotic maps between classifying spaces of the unitary groups $BU_n \rightarrow BU_m$. If $n > 18$ and $m \leq t(n) := \frac{1}{2}n(n-1)(n+2)$ we obtain certain classification of such maps.

Let $G$ be a compact connected Lie group. A (homotopy class) of a map $f: BG \rightarrow BU_m$ will be called a homotopy representation of the group $G$. For a prime $p$, a $p$-toral group is a group whose identity component is a torus and group of connected components is a finite $p$-group. According to the Dwyer-Zabrodsky-Notbohm Theorem [11, 26] a homotopy representation $f$ restricted to any $p$-toral subgroup $P \subseteq G$ is induced by a group homomorphism $\rho_P: P \rightarrow U_m$, i.e. $f|_{BP} \sim B\rho_P^f$. In particular, if $P = T_G \subseteq G$ is a maximal torus we obtain a representation $\rho^f_{T_G}: T_G \rightarrow U_m$ which is invariant (up to an isomorphism) under the action of the Weyl group $W_G := N_G(T_G)/T_G$ on $T_G$. Thus the homotopy representation $f$ defines a $W_G$-invariant element in $R(T_G)$ – the representation ring of the torus. Since the restriction homomorphism $\text{res}_{T_G}^G: R(G) \rightarrow R(T)^W_G$ is an isomorphism [5], we can associate to every homotopy representation $f$ its character $\rho^f \in R(G)$. The construction obviously generalizes the construction of a character of a linear representation.

The main question is: what characters $\mu \in R(G)$ are homotopy characters? The Dwyer-Zabrodsky-Notbohm Theorem implies that for any $p$-toral subgroup $P$ restriction $\mu|_P \in R^+(P)$ must be a genuine representation of $P$. Characters of $G$ having such property we will call $p$-characters, and those which are $p$-characters for every prime $p$ will be called $P$-characters. Thus the first step in classification of maps $BU_n \rightarrow BU_m$ is a characterisation of $P$-characters of $U_n$. This purely algebraic question is considered in Chapters 1-4.

We begin Chapter 1 with recalling basic definitions in representation theory and then define $p$-characters and prove their elementary properties. In Section 1.5 we define a slant-product in representation ring of a product of two groups which, in subsequent sections, is used for reduction from larger to smaller subgroups. In some special cases the operation is called a reduction of a character.

In Chapter 2 we apply reduction of characters to formulate criteria when a virtual character of a unitary group is a $p$-character. For that we need a careful description of the maximal $p$-toral subgroup of $U_n$ as an iterated wreath product of one-dimensional torus and cyclic groups (Sec. 2.2) and study its representations (Sec. 2.3). In Sec. 2.4 the main characterization theorems of $p$-characters of the unitary groups [??], [??] are proved. Detailed study of the case of $U_p$ is carried on in Sec. 2.5 resulting in a simple characterization of $p$-characters of $U_p$ [??]. In Sec. 2.6 we describe a group endomorphism of the maximal $p$-toral subgroup $N^n_p \subseteq U_n$ which defines the Adams operation $\Psi^k: R(N^n_p) \rightarrow R(N^n_p)$. In general, effect of the $k$-th Adams operation on a character we call its $k$-twisting.

In Chapter 3 we construct families of $P$-characters of $U_n$. In Sect. 3.1 we describe some representations of $U_n$ which are used for construction of $p$-characters of $U_n$. In Section 3.2 we list candidates for $P$-characters and check which of them actually are $P$-characters. In last
Section 3.3 we describe examples showing that a decomposition of $p$-characters into sum of the indecomposable $p$-characters is not unique.

In Chapter 4 we show that for unitary group $U_n$ such that $n > 18$, $P$-characters of dimension $\leq t(n)$ are exactly the ones constructed in Chapter 3. A proof relies on a careful analysis of dimensions of the symmetrized weights of the torus which can occur in decompositions of $P$-characters restricted to the maximal torus. Proof of the main algebraic result is presented in Section 4.4.

The last Chapter 5 is devoted to topological application of the algebraic result which occupies Chapters 1-4. The main result says that all $P$-characters listed in Section 4.4 are indeed homotopy characters. The key idea is a splitting property of characters recalled in Sec. 5.2. We say that a (homotopy) character $\nu \in R(G)$ has a splitting property if any character $\chi$ such that $\chi + \mu$ is a homotopy character is also a homotopy character. Results of [19], and a recent paper [21] imply that the trivial character and characters of the Adams operations have splitting property. We are lucky since the results, combined with other results of [15], suffice to prove that all our $P$-characters are indeed homotopy characters of $U_n$.

All the groups considered in this dissertation are compact Lie; all subgroups considered are closed. We denote by $Grp$ the category of compact Lie groups and their homomorphisms.
Literatura


