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REPORT
on PhD thesis of **Yixiao Qiao** entitled
"Growth parameters and chaos in discrete dynamics"

PhD thesis of Yixiao Qiao consists of 87 pages and is divided into 5 main sections. It contains bibliography, abstract, acknowledgments and short introduction presenting contents of the thesis. Thesis was written under supervision of Yonatan Gutman (IMPAN) and Wen Huang and Xiangdong Ye (USTC). The thesis is written in English. Core part of the thesis are 4 articles (co)authored by Yixiao Qiao:

[1] Y. Qiao, *Topological complexity, minimality and systems of order two on torus*, Science China Mathematics, **59** (2016), 503–514.
[2] Y. Qiao and X. Zhou. *Zero sequence entropy and entropy dimension*, Discrete and Continuous Dynamical Systems, **37** (2017), 435448.
[3] J. Li and Y. Qiao, *Mean Li-Yorke chaos along some good sequences*, Monatshefte für Mathematik, to appear.
[4] Y. Gutman, Y. Qiao, and G. Szabó, *The embedding problem in topological dynamics and Takens theorem*, Nonlinearity, to appear.

From strict point of view all these topics are independent, but to some extent bond together by such notions as topological entropy or various types of dimension. The results also share joint toolbox provided by topological dynamics and ergodic theory. From that point of view, such combination is acceptable and can be considered as devoted to a single wider topic. Let me now present contents of the thesis in more detail.

Chapter 1 has a preliminary character. It collects most of main definitions and concepts used later in the thesis. It contains notions from ergodic theory, such as invariant measures, measure-theoretic and sequence entropy, Pinsker σ-algebra, conditional expectation or entropy dimension; notions from topological dynamics such as equicontinuity, topological entropy, mean dimension or Mycielski theorem or Li-Yorke chaos.

In Chapter 2 the main theme is put on construction of some special type of skew products on the tori over an irrational rotation, that is dynamical systems of the form

$$T\colon \mathbb{T}^2 \ni (x, y) \mapsto (x + \alpha, f(x) + y) \in \mathbb{T}^2$$

where f is a properly chosen map on the unit circle. The first aim is to construct this map in such a way that the complexity grows in a linear manner, that is

$$c_1(\varepsilon)n \leq r(n, \varepsilon) \leq c_2(\varepsilon)n$$

where $r(n, \varepsilon)$ denotes minimal cardinality of (n, ε)-spanning set for T. It is proven that if f has bounded variation then this condition is satisfied. As the next step, the author provides conditions on f which ensure that T is not a system of order 2. Final step is a clever and nontrivial construction combining these two ingredients together (i.e. construction of a special function of bounded variation). This way the author is able to answer in the negative a part of question from 2014 stated by Host, Kra and Maass. Strictly speaking, the author shows that there are distal systems with low complexity which are not 2-step nilsystems. The argument in this chapter is very nice, and the construction is very ingenious. It also shows that the author is well informed in this topic, since there are many reductions changing the original question into a more accessible equivalent statement. To achieve this, it is necessary to apply a few advanced recent results by various authors.

Chapter 3 studies relations between sequence entropy of dynamical system (X,T) and associated system $(M(X),T)$ induced by it on the space of all probability measures. It was known for a while that both systems have zero entropy at the same time, and that both are null at the same time. The author makes a step further, proving that if S is a sequence along which one of the systems (X,T), $(M(X),T)$ has zero sequence entropy, then the same must hold for the second system. She also generalizes this result obtaining conditions for sequence entropies of an ergodic measure μ and associated collection of quasi-factors $Q(\mu)$. Second goal of this chapter are results of similar flavor relating topological entropy dimension of both systems (both in topological and measure-theoretic setting).

In Chapter 4 the author is interested in a version of distributional chaos DC2 (known also as *mean Li-Yorke chaos*). A few years ago Downarowicz solved an old open problem, proving that positive topological entropy implies DC2. In this chapter n-tuples instead of pairs are considered, and additionally densities are calculated along some sequences instead of $\mathbb{N}$. The main notion here are so-called good sequences and elements of fibers of Pinsker factor. The author proves that for many fibers we can find a Cantor set reflecting chaos along a good sequence, provided that Pinsker σ-algebra of measure μ (of positive entropy) is characteristic for this sequence. Methodology here is different compared to Downarowicz original argument, and relies on technique developed later by Huang, Li and Ye. The chapter ends with similar result for non-invertible maps by delicate analysis of dynamics of associated shift homeomorphism (natural extension).

The final chapter of the thesis studies problems of embedding dynamics in d-cubical shift of the Hilbert cube. These studies fit very well in the long history of research undertaken by many very good mathematicians. The main result of the chapter is a version of Takens' embedding theorem. It says that given an invertible dynamical system, the set of continuous evaluations $h\colon X \mapsto [0,1]^m$ which define embedding of (X,T) in the cube $([0,1]^m)^{2d+1}$ is generic, where $d = \dim(X)$ and lower bound on m depends on dimension of a subset of the set of periodic points. In other words, dynamics of transitive (X,T) can be well approximated by a "moving window" in time series for almost all (in topological sense) evaluations h defining this time series.

The chapter ends with considerations motivated by Lindenstrauss-Tsukamoto conjecture from 2014. While this conjecture is not solved, the author points out that its statements hold generically. It is based on two results. First of them states that there is a residual set of homeomorphism of Cantor set, such that any two homeomorphisms in that set are conjugate. Second element is a result of Hochman which states that this class represents a residual set among all subsystems of $C = ([0,1]^{\mathbb{N}})^{\mathbb{Z}}$ endowed with natural shift map. On the other hand, every dynamical system is a subsystem of this shift. This leads to an observation that typical dynamical system (in the sense of topology on C) satisfies the above mentioned conjecture.

In general, thesis is very well written and I had no problem to follow presented arguments. It was not completely self-contained, and in some cases it was necessary to have knowledge about some recent advances in the field to understand these arguments. However it was never beyond the point that can be expected from a well informed reader familiar with the topic, so these shortcomings are acceptable. Let ma present a few remarks on the mathematical content. There are only a few, since as I said before, presentation is very clear and sound:

- Theorem 1.1.21: [EW] defines conditional expectation in a slightly different way. If we bother to present the definition of conditional expectation, it would be nice to comment why these two definitions coincide. It is especially important, because the feature of conditional expectation used many times in the thesis (which is main ingredient of definition e.g. in [EW]) is that
$$\int_A E(f|\mathcal{A})d\mu = \int_A f d\mu \quad \text{for any } A \in \mathcal{A}.$$
Unfortunately, the author does not recall this property in the thesis.
- What is the reason to present complete proofs of Lemma 1.6.6. and 1.6.7? Are they independent proofs of results from [Gut15]? Is the proof of Lemma 1.6.6. induction on s? "almost surely w.r.t. Lebesgue measure" is not defined. Without this definition it is almost impossible to understand the statement (and the proof).
- There is no functions g_n, g in the statement of Lemma 2.3.1. Everywhere in the proof g_n should be f_n. In the definition of A, B points x_1, x_2 should be x. The condition we are going to show is that orbits $O(x_1, R_\alpha) \subset A$ and $O(x_2, R_\alpha) \subset B$ proving density of these sets that way (and it is the place where x_1, x_2 appear). Additionally it should be something like $A = \{y \in \mathbb{T}^1 ...\}$ or $\int_0^1 f(y)dy$ to avoid collision.
- 51^2: Definition of isomorphism (section 1.1.7) says that ϕ should be invertible (strong isomorphism). But ϕ here is not onto, so by what method can we construct the isomorphism (whose existence is announced to be easy to verify)? Of course, we can use Theorem 1.2.2. with a weaker property which ϕ has.
- 61^6: "By a similar argument as in [HYY14,Lemma 2.4]..." - the argument seems standard and not very hard. It would be good to present it in detail, instead of referring to an external paper. Without this, the proof is not self-contained which is small disadvantage. The same comment applies to 62^3 and 63_6.
- 66^{22}: "the first named author" - do you mean Yonatan Gutman?
- 69^5: By standard notation $g \times f$ denotes $(g(x), f(y))$ while usually (g, f) is used to denote $(g(x), f(x))$. It is clear that this second meaning is used here.

As the last part of this report, I will present a few remarks on editorial aspects of the thesis. They have no impact on my judgment of mathematical correctness of the thesis, but I hope that the author will find them helpful in her future work. It is very much visible that the thesis is a combination of different papers (sometimes with completely different notation), which caused many problems. Unfortunately they were not completely removed in the editorial process.

- 17_{14}: $\mathbb{Z}_+$ is not a group, while it is demanded by the definition a few lines above.
- 17 : I would rather write "discrete semi-dynamical system" instead of "semi-discrete dynamical system".
- 17_8: topological dynamical system was defined earlier.
- 17_6: definition of semi-discrete TDS does not coincide with earlier definition (we additionally assume suriectivity here)
- 18^1: formally we need particular metric in (3), (4) or we should say earlier in the definition that X is metrizable instead of requiring that it is a metric space.
- 18_8: what is "d-step topologically nilpotent"?
- 18_7: Do we assume anything about G in the sentence "Let $\tau \in G$ and T.... Then... d-step nilsystem"?
- 18_4: Personally, I would change the order in this sentence to "closure of ... in X^X endowed ..."
- 19^{10}: How is exactly "inverse limit of d-step nilsystems" defined?

- Definition of "good sequence": is our intention to allow nondecreasing or even "rare random element" in these sequences? Is there any good motivation for that?
- In my opinion Theorem 1.1.13 should say something like: $\mathbb{N}$ regarded as an increasing sequence is very good. "The" in the beginning of the statement suggests this meaning, but it is not explicit.
- 22^6: What are abelian extensions and (return time) Morse sequence?
- 23_{13}: $\alpha(x)$ is not defined so far; the definition appears much later in the thesis.
- 24_{14}: "It is well known..." is not completely obvious, even if similar to "standard" entropy. A reference would be appreciated here. Also after Theorem 1.2.1 it would be worthy to comment that sequence entropy does not behave under $n-1$ factors as nice as topological entropy does. While some results look similar, some other are completely different, which makes these topics hard and provides additional motivation.
- 26^2: should be $\bigcap_{k=0}^{\infty}$
- 26^5: $W^s(x,T)$ and $W^u(x,T)$ (i.e. x instead of X)
- 27^{10}: ,,dose" should be ,,does"
- 30_{13}: "In 2014..." is exact repetition of Theorem 1.5.2 below.
- 30_3: why $a_n \in \mathbb{N}$ for all $n \geq 1$ in the definition, i.e. why all $a_n > 0$? Don't we allow $\mathbb{Q}$ on purpose? In that case the first sentence in Proposition 1.6.1(2) seems unnecessary.
- 31^{15}: ,,It is well known that the set..." - a reference for this statement would be appreciated.
- in definition of $v(\alpha)$ it is worth recalling that $||n\alpha||$ was defined in Example 1.1.5.
- It would be nice to stress in Theorem 1.6.5. that d_u is this complete metric.
- 36^{15}: "Recall that $T : \ldots$" should rather say that we fix this notation for the reminder of this section.
- 36^{16}: "Notice that $v(n) =$..." - it is definition of $v(n)$ isn't it? Nothing to observe there.
- 37^{12}: "Take $M =$" should rather be: where $M =$.
- 39^{12}: constant l was already used to denote degree of f.
- 41_2: Here we define g not f, so better flip it to $g(x) = f(x) - l$.
- Theorem 2.3.2: Sometimes we do calculations in $\mathbb{R}$ and sometimes in $[0,1)$ without distinguishing between them. It would be elegant to emphasize that the second case occurs by writing " mod 1" or using some other notation (e.g. $I \subset J + \mathbb{Z}$ for sets inclusions).
- 46_6: Since $m(\{\alpha \ldots\})$ - I would appreciate comment similar to the one on p. 31 here.
- 47_2: What does it mean that points (not orbits) are ε-separated?
- 49^1: $i \in \mathbb{N}$.
- Chapter 4: Notation $T^a f$ is not consistent in the paper. It denotes $T^a f := f \circ T^a$, while $T^a x$ denotes $T^a(x)$. It is also not explicitly explained.
- 57_7: "Note that" should be "Recall that".
- 58_7: From pointwise good we know only that the limit exists, so we should have $\lim_{N\to\infty} \ldots = \tilde{f}_n(x)$ instead of $\lim_{N\to\infty} \ldots = f_n(x)$.
- 60_{14}: "A n-tuple" should be "An n-tuple".
- 61^1: By Theorem 1.2.6(2) not (1).
- 65^3: "as simple as possible".
- 68_9: the standard name in topology for your ε-embedding is ε-map.
- 69^5: what does equivariant mean?

In my opinion results contained in four papers [1-4], which form the core of the thesis, provide more than enough material for good Ph.D. thesis. They were published (or accepted for publication) in journals indexed by JCR; all of these journals have good reputation in the mathematical community. Presented results are definitely nontrivial, some of the proofs involve long sequence of consecutive arguments, often divided into a few minor steps. This proves good intuition of the author, deep knowledge in the research topic and her maturity as mathematician. In my opinion Yixiao Qiao is able to conduct independently high quality research and she has sufficient experience to continue her research on topics related directly to her thesis or some other problems on combinatorial, topological or ergodic aspects of dynamical systems. In the view of the above I consider that Yixiao Qiao deserves to obtain the Ph.D. degree in mathematics. In my opinion all formal and traditional requirement for PhD theses are fully satisfied.