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Professor Piotr NOWAK
IMPAN
00-656 Warszawa
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Dear Professor NOWAK,

You have kindly asked me for an assessment of Dr Artem DUDKO's research, related to his habilitation procedure.

I have never met Artem DUDKO (AD) in person, but am quite familiar with his results, which are of fundamental importance. I recommend without reservation that he be granted an habilitation.

AD's supporting material ("Habilitation essay") is extremely well written, and of great clarity; I have little to add to it. Please allow me however to provide some context to his results, to highlight their importance to the mathematical community.

AD's strongest research results are at the juncture of group theory and analysis: given a group G, one studies it using tools from functional analysis, namely operator algebras, Banach spaces, spectral decompositions, etc.; the main goal, in this area, is to relate algebraic properties with analytical ones.

One such specific object is the *space of characters* of a group G. Recall that a character is a map $\chi\colon G \to \mathbb{C}$ that is normalized ($\chi(1) = 1$), central ($\chi(gh) = \chi(hg)$) and positive (for any $g_1, \dots, g_n \in G$ the matrix $(\chi(g_i g_j^{-1}))_{i,j=1\cdots n}$ is positive definite).

Characters are of fundamental importance in the theory of finite groups — given a finite-dimensional representation $\rho\colon G \to GL_n(\mathbb{C})$ the map $g \mapsto \operatorname{tr}(\rho(g))/n$ is a character — but also, more and more so, for infinite groups. In fact, the example above generalizes as follows: given a character χ on G, there exists a representation π_χ of G on a Hilbert space $\mathcal{H}$, and a vector $\xi \in \mathcal{H}$, such that $\chi(g) = \langle \xi, \pi_\chi(g)\xi \rangle$; the representation π_χ generates a von Neumann algebra M_χ, on which χ extends to a trace. Conversely, given any von Neumann algebra closure of G with a trace, one obtains a character by $\chi(g) = \operatorname{tr}(g\cdot)$.

Convex combinations of characters are again characters, so it makes sense to consider *irreducible* characters, namely those that cannot be decomposed as a non-trivial convex combination. This amounts, in the context of von Neumann algebras, to considering *factors*, namely von Neumann algebras whose centre is reduced to $\mathbb{C}$.

A large source of characters comes from the following construction; it is interesting in that it both provides a rich supply of characters, and offers character theory as a means of studying the original objects. Let G act on a standard probability space (X, Σ, μ) by measure-preserving transformation. Then $g \mapsto \mu(\mathrm{Fix}(g))$ is a character. Here $\mathrm{Fix}(G)$ is the set of fixed points $\{x \in X : gx = x\}$.

In a series of ground-breaking articles, Vershik studied characters of the infinite symmetric group, and established some basic results related to characters. In particular, he calls an action of a countable group G on (X, Σ, μ) *totally non-free* if the $\mathrm{Fix}(g)$ generate the σ-algebra Σ, equivalently if for almost all $x \neq y \in X$ have different stabilizers: $G_x \neq G_y$.

He then made the claim that, if (X, Σ, μ) is ergodic (i.e. X has no G-invariant subset of measure $\neq 0, 1$), then the character $\mu(\mathrm{Fix}(g))$ is irreducible if and only if the action is totally non-free.

Unfortunately, this claim is wrong, and we owe it to AD (in collaboration with R. Grigorchuk) that in fact both directions of the claim are invalid.

Not only that, they give a stronger condition than "totally non-free" on the action, namely that every measurable set A can be approximated in measure by some $\mathrm{Fix}(g)$, which *does* imply that the corresponding character is irreducible.

The example of a totally non-free action with reducible character is simply $\mathrm{Sym}(n)$ acting on $\{1, \ldots, n\}$, for $n \geq 3$; the permutational character decomposes as a combination of the trivial character and its orthogonal. This action can be brought to a non-atomic probability space by multiplying it with that of a totally non-free action on Lebesgue space.

Conversely, if G is such that all its non-trivial conjugacy classes are infinite, then its essentially free actions are indecomposable; one may e.g. take $X = U(n)$ the unitary group, and G a dense countable subgroup containing no scalar matrices except the identity.

Even if wrong, Vershik's claim still serves as an important motivating idea: relate the richness of a character of G to the richness of the action it comes from.

If it were only for these results, AD's place in mathematics would already be well established. However, he is also the author of a large number of other important contributions.

For example, with Konstantin Medynets he studied characters on inductive limits of symmetric groups; these are groups obtained as ascending unions of symmetric groups $\mathrm{Sym}(n_1), \mathrm{Sym}(n_2), \ldots$, with a prescribed series of (diagonal) inclusions of each $\mathrm{Sym}(n_i)$ into $\mathrm{Sym}(n_{i+1})$.

In another article, they introduce the notion of a *compressible* action of a group G on a topological space, and show how it is related to properties of characters of G.

Finally, another very active topic is the study of "invariant random subgroups". The idea is to consider the space $\mathrm{Sub}(G)$ of subgroups of a group G; it is naturally a subspace of 2^G, and comes naturally equipped with the structure of a Polish space, and an action of G by conjugation. Normal subgroups are naturally fixed points of this action; fixed *measures* are called "invariant random subgroups".

Given a (non-free to make it interesting) action of G on (X, Σ, μ) as above, one obtains naturally a map $X \to \mathrm{Sub}(G)$, given by point stabilizers: $x \mapsto G_x$. Pushing the measure μ through this map yields an invariant random subgroup of G. In the last paper that I have consulted, AD (again with Konstantin Medynets) studied invariant random subgroups of inductive limits of symmetric groups, and could obtain in this manner their complete classification.

Perhaps I may add a word to the fact that AD has a large number of articles in collaboration. I certainly do not think it should be counted negatively; his article [H1] was written alone, and one recognizes from the style of the other articles [H2...H7] that they were mainly authored by AD.

In summary, AD is a talented mathematician, who could solve important questions in mathematics, at the border of group theory and analysis, and embarked in coherent research project. He attracted other mathematicians to this area and developed fruitful collaborations with them. He is entirely apt to direct research through a habilitation.

With my best regards,

Laurent Bartholdi