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REPORT ON THE HABILITATION APPLICATION OF DR. MACIEJ BOCHEŃSKI

Dr. Maciej Bocheński has been an Adjunct professor at the Faculty of Mathematics and Computer Science of the University of Warmia and Mazury in Olsztyn since 2015. He was previously an Assistant professor there, after completing his PhD at the same university in 2014, under the supervision of Prof. Aleksy Tralle.

Dr. Bocheński's Habilitation thesis is based on four papers, published between 2015 and 2019 in high-level international journals (International Mathematics Research Notices, Proceedings of the American Mathematical Society, Asian Journal of Mathematics, and Experimental Mathematics). One of these papers is written alone, two are written jointly with A. Tralle, and one jointly with P. Jastrzębski, A. Szczepkowska, A. Tralle, and A. Woike.

General description of the research

Dr. Bocheński's research concerns properly discontinuous actions of discrete groups Γ on homogeneous spaces G/H where G is a real reductive Lie group and H a closed reductive subgroup of G. If the action of Γ on G/H is properly discontinuous, then the quotient $\Gamma\backslash G/H$ is an orbifold, and even a manifold if the action is free (e.g. if Γ is torsion-free — a case that we may always reduce to if Γ is finitely generated, using Selberg's lemma). Thus properly discontinuous actions on G/H yield quotient manifolds locally modelled on G/H, which naturally inherit any G-invariant geometric structure from G/H, and this explains the geometric importance of understanding such properly discontinuous actions, in an approach taking its roots in Klein's Erlangen Program.

In the case that H is compact (e.g. G/H is a Riemannian symmetric space), any discrete subgroup Γ of G acts properly discontinuously on G/H. This is far from being the case when H is noncompact. For instance, if H has the same real rank as G, then only finite groups may act properly discontinuously on G/H: this was first observed by Calabi and Markus for $G/H = \mathrm{SO}(n,1)/\mathrm{SO}(n-1,1)$ (de Sitter space), and proved in full generality by Kobayashi in the late 1980s. When H is noncompact, it is actually a difficult problem to understand when there exist "interesting" discrete subgroups Γ of G acting properly discontinuously on G/H and, in these situations, to describe these discrete groups Γ. Here "interesting" may mean several things:

(i) Γ is infinite: the criterion on G/H for the existence of such Γ is that the real rank of H must be strictly smaller than that of G (Kobayashi, Benoist);

(ii) Γ is not virtually abelian: a full criterion for the existence of such Γ was provided by Benoist; building on this, in his joint work [O1], Dr. Bocheński gave another criterion which is particularly handy when looking at examples; his criterion involves the new notion of *a-hyperbolic rank*;

(iii) Γ is contained in a subgroup L of G acting properly on G/H and which is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ (in particular, up to replacing Γ by another discrete subgroup of L, we may take it to be isomorphic to a nonabelian free group or to the fundamental group of a closed hyperbolic surface): for the existence of such $\Gamma \subset L$ a full criterion on G/H (involving the a-hyperbolic rank) was provided by Dr. Bocheński in [O2], under some *strong regularity* assumption on G/H; Okuda had previously shown that, when G/H is a semisimple symmetric space, the existence of such $\Gamma \subset L$ is equivalent to the existence of non-virtually abelian groups acting properly discontinously on G/H as in (ii), but this is not true for general G/H;

(iv) Γ is "large enough" so that the quotient $\Gamma\backslash G/H$ is compact: there is so far no full criterion on G/H for the existence of such Γ. One approach, which has given rise to a very rich literature in the past forty years, is to find various obstructions to the existence of compact quotient manifolds $\Gamma\backslash G/H$; such obstructions may be Lie-theoretic (Kobayashi, Benoist, Morita), geometric (Benoist–Labourie), dynamical (Zimmer, Labourie–Mozes–Zimmer), representation-theoretic (Margulis, Shalom), topological (Kobayashi, Kobayashi–Ono, Morita, Tholozan), etc. In [O4], Dr. Bocheński and his coauthors, building on a cohomological obstruction of Tholozan, used computer methods to obtain a list of new reductive homogeneous spaces G/H for which compact quotient manifolds $\Gamma\backslash G/H$ do not exist.

In cases where they are known to exist, Dr. Bocheński also worked on the algebraic description of "interesting" discrete subgroups Γ acting properly discontinuously on G/H. In particular, in his joint work [O3], he proved that, in certain situations, discrete groups Γ giving rise to compact quotient manifolds $\Gamma\backslash G/H$ can never be solvable.

More precise description of the scientific achievement

In his paper [O1], joint with A. Tralle, Dr. Bocheński introduces the notion of *a-hyperbolic rank* for a real reductive Lie group G, denoted by $\mathrm{rank}_{a-hyp}(G)$: this is by definition the dimension of the set of fixed points of the opposition involution in a Cartan subspace of G. Note that, given a reductive subgroup H of G, the following inequalities hold, where $\mathrm{rank}_{\mathbb{R}}$ denotes the real rank:

$$\begin{array}{ccc} \mathrm{rank}_{\mathbb{R}}(G) & \geq & \mathrm{rank}_{\mathbb{R}}(H) \\ \vee\!| & & \vee\!| \\ \mathrm{rank}_{a-hyp}(G) & & \mathrm{rank}_{a-hyp}(H). \end{array}$$

Under mild assumptions on G/H (namely, G is semisimple and H has compact center and finitely many connected components), Bocheński and Tralle prove the following:

- if $\mathrm{rank}_{a-hyp}(G) = \mathrm{rank}_{a-hyp}(H)$, then only virtually abelian discrete groups may act properly discontinuously on G/H.
- if $\mathrm{rank}_{a-hyp}(G) > \mathrm{rank}_{\mathbb{R}}(H)$, then there exist properly discontinuous actions on G/H by discrete groups which are not virtually abelian.

This result is very nice, in particular because the rank conditions involved here are easy to check in practice on examples, and because they do not depend on the embedding of H in G. This enables the authors to give new examples of reductive homogeneous spaces G/H that do not admit properly discontinuous actions by non-virtually abelian groups (hence that do not admit compact quotient manifolds), as well as new examples of G/H that do admit properly discontinuous actions by non-virtually abelian groups.

In his paper [O2], Dr. Bocheński introduces a *strong regularity* condition for reductive homogeneous spaces G/H; this condition is satisfied by several important classes of reductive homogeneous spaces (e.g. spaces of parabolic type, spaces induced by regular embeddings $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$, or certain k-symmetric spaces). He proves that, when the strong regularity condition is satisfied, the existence of properly discontinuous actions on G/H by non-virtually abelian groups (which is governed by the a-hyperbolic rank, see above) implies the existence of a closed subgroup L of G acting properly on G/H which is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$; in particular, any discrete subgroup Γ of L acts properly discontinuously on G/H, and so this yields properly discontinuous actions on G/H by nonabelian free groups and fundamental groups of closed surfaces of any genus ≥ 2, which is very interesting.

In his paper [O3], joint with A. Tralle, Dr. Bocheński considers reductive homogeneous spaces G/H where H is the semisimple part of the centralizer of some hyperbolic element of G. He proves that for any compact quotient manifold $\Gamma\backslash G/H$, the group Γ cannot be solvable. This improves a result of Benoist from 1996, which stated that Γ could not be nilpotent (without any assumption on the reductive group H). The proof goes by contradiction and reduces to Lie algebra considerations by taking the syndetic hull of a solvable discrete group Γ.

Finally, in his paper [O4], joint with P. Jastrzębski, A. Szczepkowska, A. Tralle, and A. Woike, Dr. Bocheński uses computer methods (implemented in GAP) to find new examples of reductive homogeneous spaces G/H that do not admit any compact quotient manifold $\Gamma\backslash G/H$. This work is based on a cohomological obstruction to the existence of compact quotients that was recently obtained by Tholozan. Computer methods have not been very much used so far in the theory of compact quotients of reductive homogeneous spaces, and Dr. Bocheński's joint paper [O4] shows that they can be a very powerful tool. One can hope that this will lead the way to further interesting results about properly discontinuous actions on homogeneous spaces.

Other publications

Dr. Bocheński mentions eight other publications, which have all been published in respectable journals between 2014 and 2017, with various sets of coauthors including P. Jastrzębski, M. Ogryzek, T. Okuda, A. Szczepkowska, A. Tralle, and A. Woike. These publications concern various topics, including proper actions on homogeneous spaces, fat fiber bundles, and generalized symplectic symmetric spaces.

General opinion

Dr. Bocheński's research on properly discontinuous actions on reductive homogeneous spaces lies at the heart of modern geometry and Lie theory. I find his work to be of very high quality. He has in particular obtained striking results in his paper [O1], where the notion of a-hyperbolic rank is introduced and then used to give conditions for the existence or non-existence of properly discontinuous actions by non-virtually abelian discrete groups. The papers [O2] and [O3], which provide conditions for the existence of properly discontinuous actions by free groups and surface groups, and for the non-existence of compact quotient manifolds, are also very interesting. I find the use of computer methods in [O4] quite innovative and promising. Finally, Dr. Bocheński's other publications also show that he has been a very active researcher since his PhD.

Conclusion

In my opinion, Dr. Maciej Bocheński's work and Habilitation thesis are of very high quality, and comply with all the requirements of Polish law and practice needed to confer the degree of Habilitation in Mathematics. I strongly support his application.

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