REPORT ON THE HABILITATION THESIS BY M. BOCHEŃSKI

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1. Introduction

The dissertation under review is concerned with the question as to whether there exists a discrete subgroup $\Gamma \subseteq G$ of a semisimple real Lie group G such that the natural action of Γ on the homogeneous space G/H is proper and cocompact. The latter means that the quotient space $M = \Gamma\backslash G/H$ is compact. In this case the quotient space M is called compact Clifford-Klein form.

If the subgroup H is compact then the answer is in the affirmative by the classical results on existence of cocompact lattices in semisimple Lie groups. If the subgroup H is noncompact then there is no definitive answer. That is, sometimes a suitable subgroup Γ exists and sometimes it does not. The dissertation under review is concerned with this case, that is, G/H noncompact, and significantly enhances our knowledge in this area.

2. Review of the results

Results of [O1] (joint with Tralle). The first result of the dissertation introduces a certain numerical invariant of a semisimple Lie algebra called the $\mathfrak{a}$-hyperbolic rank. Then the following theorem is proven:

Theorem (Bocheński-Tralle). *Let G be a connected semisimple linear group and let $H \subseteq G$ be a reductive subgroup with finite number of connected components. Then*

(1) *If the $\mathfrak{a}$-hyperbolic ranks of the lie algebras of G and H are equal then there there is no subgroup $\Gamma \subseteq G$ which is not virtually abelian and acts properly on G/H. Consequently G/H has no compact Clifford-Klein form.*

(2) *If the $\mathfrak{a}$-hyperbolic rank of $\mathfrak{g}$ is bigger than the real rank of $\mathfrak{h}$ then G/H admits a proper action of a group $\Gamma \subseteq G$ which is not virtually abelian.*

Both the definition of the $\mathfrak{a}$-hyperbolic rank and the proof of the above theorem originate from the work of Benoist in which he proved a criterion for the existence of a proper action of Γ which is not virtually abelian on G/H. The proof is a careful translation and application of Benoist's criterion. It shows a great expertise of the authors in Lie theory.

The authors also compute $\mathfrak{a}$-hyperbolic ranks of simple Lie algebras and use these computations to apply their theorem in constructing new examples. They provide many new examples of homogeneous spaces G/H not admitting compact Clifford-Klein forms as well as examples of homogeneous spaces admitting proper actions of discrete groups $\Gamma \subset G$ which are not virtually abelian.

Results of [O2] (solo). The main result here is concerned with understanding of the following conditions under the assumption that G/H is *strongly regular* which is a condition on the relationship between the split Cartan subalgebras of $\mathfrak{g}$ and of $[\mathfrak{h}, \mathfrak{h}]$:

(1) G/H admits a properly discontinuous action of an infinite subgroup of G;

(2) G/H admits a properly discontinuous action of an infinite subgroup of G which is not virtually abelian;

(3) G/H admits a properly discontinuous action of a subgroup $L \subseteq G$ which is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$;

Comment. *I think that it should be mentioned in (2) that the subgroup of G is discrete, since Benoist's criterion is used in the proof.*

The main result states that if the $\mathrm{rank}_{\mathfrak{a}-\mathrm{hyp}}(\mathfrak{h}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{h})$ then the conditions (2) and (3) are equivalent to the condition $\mathrm{rank}_{\mathfrak{a}-\mathrm{hyp}}(\mathfrak{g}) > \mathrm{rank}_{\mathfrak{a}-\mathrm{hyp}}(\mathfrak{h})$. As a corollary Bocheński obtained the following necessary condition for the existence of a compact Clifford-Klein form.

Corollary (Bocheński). *Let G/H be a strongly regular homogeneous space such that* $\mathrm{rank}_{\mathfrak{a}-\mathrm{hyp}}(\mathfrak{h}) = \mathrm{rank}_{\mathbb{R}}(\mathfrak{h})$. *Then if G/H admits a compact Clifford-Klein form then G/H admits a properly discontinuous action of $L \subset G$ which is locally isomorphic to* $\mathrm{SL}(2, \mathbb{R})$.

Finally he applies the results to provide a new example of a homogeneous space admitting (2) but not (3).

Results of [O3] (joint with Tralle). The main result states that in a compact Clifford-Klein form $\Gamma\backslash G/H$ the subgroup Γ cannot be virtually solvable. Here G is assumed to be linear and semisimple of non-compact type plus a certain technical condition on H. For example, the result applies to $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$.

Results of [O4] (joint with Jastrzębski, Szczepkowska, Tralle, Woike). The starting point of the paper is a condition necessary for the existence of compact Clifford-Klein forms of G/H proven by Tholozan. It is stated in terms of cohomology of the dual symmetric space G_U/K and very hard to verify in practice. The main result of [O4] is an algorithm which checks the above condition. The implementation of the algorithm provided a list of (also new) examples of homogeneous spaces G/H which do not admit compact Clifford-Klein forms.

Also the authors proved that $G/\mathrm{SL}(2,\mathbb{R})$ does not have compact Clifford-Klein forms for a semisimple connected linear G. This generalises results of Benoist, Labourie, Margulis, Shalom and Zimmer.

3. Evaluation

I am impressed with the work of Bocheński.

First of all, it was very interesting to read what his coauthors said about his contribution to the joint papers. Based on their statement Bocheński was a genuine leader of the research group. Of course, when reading the papers and the proofs, the contribution of Tralle should not be underestimated.

Bocheński shows a great deal of expertise in Lie theory. Essentially, the core of all the papers [O1,O2,O3] are (often intricate) arguments with Lie algebras. However, he is not restricted to one type of techniques. There are differential algebraic and cohomological arguments in [O4], not to mention the computational excursion in the same paper.

The results are very concrete and he provides many new examples which I like a lot. Each paper separately is solid piece work and none of them is a spectacular piece of work. When reading them I did not feel surprised. Also, as a non-expert in Lie theory, I don't think that I fully appreciated the concept of the $\mathfrak{a}$-hyperbolic rank. However, when taking all the papers together I see an very good contribution to the subject with many concrete problems solved.

The dissertation also demonstrates the depth and breadth of the general knowledge of the subject of Clifford-Klein forms. I don't think it will be an overstatement to call him a world class expert in this area.

I recommend awarding Bocheński the *Habilitation* degree.

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