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Report on the habilitation thesis of Maciej Dolega

Maciej Dolega has made many very interesting contributions to combinatorics, especially in the theory of symmetric functions and their relation to the enumeration of maps.

He has chosen to present, for his habilitation thesis, a series of papers, all published in leading mathematical journals, on the so-called "$b-conjecture$". This conjecture, formulated by Goulden and Jackson in the 90's, relates Jack symmetric functions to the combinatorics of maps. It generalizes well known identites relating Schur functions, maps and factorizations of permutations, which have been progressively understood and are now well known, for example they are described in details in a very interesting book by Lando and Zvonkin on the subject of graphs drawn on surfaces. The substance of this conjecture is that some naturally defined functions of a parameter b (here $b = \alpha - 1$ where α is the parameter of the Jack functions) turn out to be polynomials with non-negative coefficients. The definition of these functions, as coefficients in the expansion of a the logarithm of a natural series built from Jack functions, only implies directly that they are rational functions with rational coefficients implying that, if the conjecture is true, there is some nontrivial mathematical structure hidden behind these functions.

The fact that there exists a connection between Schur functions, permutation factorizations and maps is easy to understand from an algebraic point of view since factorizations of permutations can be enumerated using symmetric groups characters and these characters appear naturally as the transition matrix between Schur function and power sum symmetric functions, while permutation factorizations can be directly represented in a combinatorial and geometric way by maps (i.e. graphs drawn on a two-dimensional surface). The very surprising fact about the $b-conjecture$ is that it relates maps to Jack polynomials which, although they are generalizations of Schur functions, have no direct relationship with representations of symmetric groups. In fact there are two values of the Jack parameter for which there is a representation theoretic interpretation of the Jack functions and these cases hint at the fact that there should be a connection between the $b-conjecture$ and orientability of surfaces. Understanding this conjecture and hopefully establishing it is therefore a challenging problem in symmetric function theory. The first contribution in the habilitation has been obtained in a joint paper with Valentin Féray where they prove the polynomiality in b. The full conjecture also asserts that the coefficients of these polynomials have nonnegative coefficients, so their result is a first important step but the full conjecture is still open. The main tool in this paper is a notion of cumulants, brought from probability theory, the cumulants being the coefficients in the expansion of the logarithm of a certain partition function. The authors prove that the polynomiality property is equivalent to a certain vanishing property of the cumulants, this vanishing property being proved by a highly nontrivial combination of algebraic and combinatorial arguments. This is major new idea in this field and it proves very fruitful. The second contribution is a paper by Maciej Dolega where he proves a partial result towards the $b-conjecture$: an explicit combinatorial formula for the top degree coefficients which involves a certain invariant η of maps. The next result is in a joint paper with Guillaume Chapuy. It is again motivated by the $b-$*conjecture* and it gives a way to put in bijective correspondance certain quadrangulations and unicellular maps. This generalizes well known constructions, going back to Schaeffer, to the case of arbitrary surfaces (i.e. not necessarily orientable). Another contribution presented in the habilitation is an application of the ideas about cumulants to Macdonald polynomials. These polynomials are a generalization of Schur polynomials and contain the Jack symmetric functions as limit case. It is natural to try to apply the ideas developped for Jack symmetric functions to this case. This is what Maciej Dolega does and he proves the analogue of the vanishing property of the cumulants used in the paper with Féray. Finally he gives another proof of this property in another paper, the last contribution presented for the habilitation, deducing it from a combinatorial formula for the Macdonald cumulants. This combinatorial formula generalizes well known work of Haglund, Haiman and Loehr and relates the Macdonald cumulants to the theory of Tutte polynomials, a very important topic in graph theory.

In summary Maciej Dolega has made important contributions towards the proof of the $b-$*conjecture*. He has displayed an amazing combinatorial strength as well as shown deep insights into the nature of the problems he has studied.

I have no doubts that this work qualifies him for an Habilitation and I support strongly his nomination to this scientific degree.