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Report

on the Habilitation thesis

ARITHMETIC OF PICARD - FUCHS DIFFERENTIAL EQUATIONS

of Mariia Vlasenko

Dear Dr. hab. Piotr Nowak, Scientific Director,

it is with great pleasure that I take upon me to write on the work of Mariia Vlasenko that she presented for her habilitation. I was asked to do so already in september of 2019, but due to certain unexpected personal circumstances, followed by the current corona crisis, I have so far not been able to do so and I am sorry for any inconveniences that may have resulted from these circumstances.

The work of Vlasenko spans a large range of topics in pure mathematics and resulted in about 20 publications. Initially she worked in probability theory and stochastics, later on problems of representations of algebras, but around 2006 her focus started shifting in the direction of number theory, with analytic methods and now it seems that her main interests focus around explicit control over finer motivic information like L-functions, special values, modular forms attached to well defined combinatorial objects like modular forms or Laurent polynomials attached to convex polytopes and motivic differential equations, using (p-adic) analytic and algebraic techniques. The papers from this last phase of her work make up the body of the habilitation.

Below I will review these papers.

Linear Mahler measures and double L-values of modular forms (2014) (with Shinder)

The Mahler measure $m(P)$ of a Laurent-polynomial P in n variables is defined as

$$m(P) = \frac{1}{(2\pi i)^n} \int_T \log |P(x_1, x_2, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n},$$

where T denotess the n-torus defined by the equations $|x_i| = 1$, has been of interest for a long time. For special polynomials these numbers tend to be expressible as special L-functions of modular forms and conjectural identifications abound, but proofs are often missing. In this paper the focus is on the linear polynomials

$$1 + x_1 + x_2 + \ldots + x_n,$$

in particular the case $n = 4$. There is a famous conjectures by Rodrigues-Villegas, which expresses $m(1 + x_1 + x_2 + x_3 + x_4)$ in terms of the special L-value of an explicit weight 3 cusp form at 4.

In the paper the Vlasenko and her coauthor Shinder express this number as a combination of *double L-values* associated to explicit modular forms. The Mahler-measure can be related to the analytic continuation of a specific solution of the Picard-Fuchs equation of the family

$$(1 + x_1 + \ldots + x_n)(1 + \frac{1}{x_1} + \ldots + \frac{1}{x_n}) = \frac{1}{t},$$

These differential equations have been studied by Verrill and have appeared in many different contexts and are great current interest. For $n = 2$ and $n = 3$ the operator is known to admit a modular parametrisation, but not so for $n \geq 4$. Vlasenko and her coauthor use that fact that by 'solving for the last variable', the Mahler-measure for $n = 4$ can be related to the continuation of an inhomogenous solution of the third order operator appearing for $n = 3$, which leads in the end, after an impressive show of analytic skill, to the double L-value expression. To my knowledge, the conjecture of Rodriguez-Villegas is still open and remains a great challenge in the Mahler-measure community. The paper is partly expository, as it also explains in detail the known cases $n = 2$ and $n = 3$, which make the paper enjoyable reading.

Equations D3 and spectral elliptic curves (2015) (with Golyshev)

Differential equations of type DN are certain Nth order operators defined as a certain determinant introduced by Golyshev and Stienstra. These operators, which depend on certain parameters, have a point of maximal unipotent monodromy at ∞ and $N+1$ further points at which the monodromy is a (pseudo) reflection. The second order operators studied by Beukers and Zagier in the wake of Apérys irrationality proofs of $\zeta(2)$ and $\zeta(3)$, are operators of geometric origin the are of type $D2$. To be more precise, these operators are Picard-Fuchs operators that belong to *modular families of elliptic curves* and presumably these are essentially all motivic operators of type $D2$. Similarly there are operators of type D3 of geometric origin, coming from the Picard-Fuchs equation of the trancendental part of the cohomology of certain families of K3-surfaces with Picard number 19, which can be related to those of type D2 by a twisted square operation and thus are still under control of elliptic modular forms.

In the paper Vlasenko and her coauthor associate to every (nondegenerate) D2 and D3 operator a specific modular form and a so-called spectral elliptic curve, depending on the full set of parameters of the operator. The main theorem now states that if the modular form for is a weight two newform, then it must coincide with the L-function of the spectral elliptic curve. For this reselt they use the Atkins-Swinnerton-Dyer congruences. As a corollary one has the multiplicativity of the coefficients of the modular form, which can be used to find equations on the parameters appearing in the operator that are necessary to enforce modularity in this specific way. By analysing these multiplicativity equations, they recover the well-known motivic equations of type D2 and D3. This is a beautiful, well-written paper that is a joy to read.

Dwork's congruences for the constant term of powers of a Laurent polynomial (2016) (with Mellit)

It is well-known that the binomial coefficients satisfy many interesting congruences, like the Lucas-congruence

$$\binom{n}{m} = \prod_{k=1}^{r} \binom{n_k}{m_k} \mod p,$$

where the n_k, m_k are the p-adic digits of n and m. Dwork has generalised these congruences to numbers b_n consisiting of general products of factorials, and used it for the explicit p-adic continuation of the associated hypergeometric function

$$f(X)=\sum_{n=0}^{\infty} b_n X^n$$

by showing the identity

$$\frac{f(X)}{f(X^p)} \equiv \frac{f_s(X)}{f_{s-1}(X)} \mod p^s \mathbf{Z}_p[[X]],$$

where

$$f_s(X)=\sum_{n=0}^{p^s-1} b_n X^n$$

denotes the s-trunction of the power series f.
Together with SAMOL I had discovered that such congruences seemed to hold for the b_n equal to the constant term of the n-th power of a Laurent polynomial P which have 0 as its single interior point, and we proved a corresponding theorem. Such constant terms b_n are much more general than products of factorials and consists basically of all *binomial sums* and include the famous *Apéry numbers*

$$b_n=\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}$$

The proof of VLASENKO and MELLIT of this theorem is however much more elegant and goes much deeper to the root of these remarkable properties. A key role was played by a certain transformation based on concatenation of p-adic digits of numbers found earlier by MELLIT, which lend itself to further generalisations and played an important role in the subsequent papers.

Formal groups and Congruences (2018)
A (1-dimensional commutative) formal group over a ring R is a power-series called *formal group law* $F(x,y) \in R[[x,y]]$ that satisifes corresponding group axioms. If $R \subset R \otimes \mathbf{Q}$, there always exists a *logarithm* for the formal group F, i.e. a power-series

$$f=\sum_{n=1} \frac{b_{n-1}}{n} x^n \in R \otimes \mathbf{Q}[[x]]$$

such that

$$F(x,y)=f(f^{-1}(x)+f^{-1}(y)).$$

In going from F to f denominators are introduced and the handle their arithmetical relationship is one of the core concerns of the theory.

Such formal groups play a role in different parts of mathematics, in particular number-theory, arithmetic geometry and algebraic topology and is by now a highly developed subject. Many important ideas and examples are scattered in the work of Honda, Stienstra, Katz, Artin and Mazur, Atkins and Swinnerton-Dyer. The bible in the field is the book by Hazewinkel from 1978 contained a fairly complete account of the theory.

Nevertheless, the paper of Vlasenko contains some useful new results, which is rather remarkable. Using the notion of p-sequence transform that developed out of the work on the Dwork-congruences, she give in *Theorem 1* a very simple congruence criterion

$$c_{mp^k-1} \in p^k R$$

for the formal group F to be p-adic integral in terms the p-sequence transformm $\{c_n\}$ of coefficients $\{b_n\}$ of a given logarithm. This result was apparently suggested by Delaygue. Although in the end it can be considered just an application of *Hazewinkels functional equation lemma*, this is a very useful fact. In *Theorem 2* Vlasenko formulates the relation between the characteristic polynomial for a formal group of height h and the coefficients of its logarithm in a particular concrete and useful way. Furthermore, Vlasenko describes the the local Euler factors of a formal L-function in terms of p-sequences. Finally, she revisits examples of Honda on hypergeometric group-laws and Stienstra on Artin-Maur group laws, in the light of the Dwork-congruences.

This is a beautiful paper. It gives an excellent and coherent account of a very classical circle of ideas, adding some new technical ideas, but keeping the account completely elementary and self-contained. It will be of great use as a quick introduction to the field.

Higher Hasse-Witt matrices (2018)
Hasse-Witt matrices appear in the description of the unit-root part of the p-adic cohomology of a variety and generalise the unit-root for an (ordinary) elliptic curve as studied by Dwork, who gave a famous formula for the unit-root in terms of the period function of the elliptic curve. For Laurent polynomials f attached to polytopes with a single interior point there is a direct generalisation of this method, which was my initial motivation to look at the Dwork-congruences in this context. Vlasenko formulated a conjecture that extends this to the case of g internal points, in which case

one can form a $g \times g$-matrices $\alpha_1, \alpha_2, \ldots, \alpha_s, \ldots$, where

$$(\alpha_s)_{u,v \in J} = \text{coefficient of } \; x^{p^s v - u} \; \text{in } \; f(x)^{p^s - 1}.$$

VLASENKO proves a *matricial version* of the DWORK-congruences and proves the existence of the p-adic limit

$$F := \lim_{s \to \infty} \alpha_{s+1} \cdot \sigma(\alpha_s)^{-1},$$

and made the conjecture that if (α_1) is invertible, than F describes the action of Frobenius operator on the unit-root chrystal of f. This was proven by A. HUANG, B. LIAN and S.-T. YAU, but it was clear from the outset that any explicit description of p-acid cohomology, like DWORK-cohomology, would lead to a natural proof. Later BEUKERS and VLASENKO gave such a self-contained, independent proof.

Conclusion

I think the overall work of VLASENKO at this stage of carrier is already quit impressive. One of the strenghts of her work is that she can isolate core problems, remove irrelevant context, dig deeper and find an elementary, self-contained road towards its proof. The arguments in the papers are always transparant, no technical details are imported or hidden. VLASENKO has written joint papers with several mathematicians of high international esteem, of which I only mention K. BRINGMANN, S. BLOCH, F. BEUKERS, V. GOLYSHEV, D. ZAGIER, S. ZWEGERS. This shows here excellent networking abilities and underlines that she is by now well-established in a certain community centered around the broad subject that might be described as *motivic algebraic geometry*, or maybe alternatively as *explicit motivic combinatorics.* She has given invited talks at many conferences, and the talks at which I was present were always of great clarity and precision. She has succesfully coorganised several conferences and programs, most notably `Hypergeometric motives and Calabi-Yau differential equations` at the `MATRIX` research Institute in Creswick and the Simons semester `Varieties: Arithmetic and Transformations` in Warsaw. She has by now ample teaching experience originating from her time in Dublin. She has a quick, systematic and precise mind and it has always stimulating to discuss with her.

So for these reasons, I strongly recommend that the works presented for the habilitation thesis should be accepted.

Duco van Straten,

Mainz, june 2020.