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Opinia w sprawie oceny osiągnięcia habiltacyjnego dra Panayotisa Smyrnelisa

Dr Smyrnelis since his graduation with the PhD degree in 2012 has published nineteen articles and co-authored a book. These papers appeared in good or very good journals. This is a remarkable achievement.

Let me comment on the content of the papers presented as the habilitation achievement. Article [S1] is devoted to a very natural generalization of a textbook problem

$$u'' = W'(u) \tag{1}$$

to the vectorial setting when W is a multiple well potential. In other words, in [S1] the authors consider the system

$$u'' = \nabla W(u), \tag{2}$$

where $W : \mathbb{R}^n \to \mathbb{R}_+$. In fact, in this case, when the zero set of W is large, defining the notion of heteroclinic or homoclinic solutions to (2) is not obvious.

The motivation to study homoclicnic, heteroclinic or periodic solutions to (2) comes from the analysis of steady states of the Ginzburg-Landau system, where the zero set of W is the unit sphere. Existence of such solutions to (2) is obtained with variational tools. This is particularly interesting, because a 'renormalizing argument' has to be performed due the the fact that the action functional should be defined over the real line.

Article [S2] is a follow up paper and dr Smyrnelis is its sole author. He uses the same variational methods, as in [S1], to establish existence of homoclinic or heteroclinic solutions to

$$\frac{d^4u}{dx^4} + W_u(u,u') - W_{uv}(u,u')u' - W_{vv}(u,u')u'' = 0, \tag{3}$$

where $W : \mathbb{R}^m \times \mathbb{R}^m \to [0,\infty)$ is smooth and $W(\cdot,0)$ is a double well potential. A motivation to study this kind of problems is the Extended Fisher-Kolmogorov equation. The author not only shows existence of solutions, but also he proves their convergence to steady states as $|x| \to \infty$. The author points to a major difference between the second order problems like (1) and (3). Namely, the latter one admits existence of 'pulses'.

Paper [S3] takes the present line of research to the next stage. It combines the variational methods with fine analysis of properties of ODEs. The article is concerned with existence of minimizers of the functional,

$$E(u) = \int_{\mathbb{R}} \left(\frac{\epsilon}{2}|u'(x)|^2 - \frac{1}{2\epsilon}\mu(x)|u|^2 + \frac{1}{4\epsilon}|u(x)|^4 - af(x)u(x) \right) dx$$

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for $u \in H^1(\mathbb{R})$. Here, $a \geq 0$ is fixed, $\epsilon \ll 1$. It is important to mention the key assumptions on μ and f. Namely, $\mu \in C^1$ is an even bounded function, $\mu'(x) < 0$ for $x > 0$ and ξ is its unique positive zero. Function f is positive for $x > 0$, odd and $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$.

The authors show existence of v, a minimizer of E. Moreover, v is of class C^2 and it satisfies the equation,

$$\epsilon^2 v''(x) + \mu(x)v(x) - v^3(x) + \epsilon a f(x) = 0.$$

Equation of this sort appears in a model of light-matter interaction in nematic liquid crystals. More specifically, this equation describes the profile of shadow kinks, when $\alpha \neq 0$. They could be observed near the boundary of the illuminated region in liquid crystals.

Interestingly, the study of the solution profile near ξ leads to the second Painlevé equation,

$$y''(s) - sy(s) - 2y^3(s) - \alpha = 0, \qquad s \in \mathbb{R}. \tag{4}$$

The main effort is on showing the asymptotics of v. We have two types of results of this sort. The first one deals with the profile of v near its zero for small ϵ. Its description exploits properties of solutions to the Painlevé equation.

The second type of asymptotics deals with the behavior of solutions to the Painlevé equation for either large positive or large negative arguments.

The authors of [S4] pick up the issue existence of entire solutions to the vectorial Allen-Cahn equation,

$$\Delta u - W_u(u) = 0, \qquad x \in \mathbb{R}^n, \tag{5}$$

where $W \geq 0$ is a smooth multi-well potential. In general solutions to (5) are not fully described even in the scalar case. The main point here is showing existence of a solution which is invariant with respect to a specific groups of reflections. Such a topic is not studied frequently in the PDE community.

The authors consider either finite or discrete reflection groups. Let me stress that setting up the tools requires a lot of effort. Thus, conveying precise results in the report is almost impossible. There are two major results in this paper. The first one guarantees, in case of a finite group of reflections, existence of nontrivial invariant (in fact f-equivariant) solutions to (5) converging, as $|x| \to \infty$, to a minimizer of W. The notion of f-equivariance is related to the fact that a group G acts on the domain while its homomorphic image transforms the target space.

The second result is similar but for discrete groups of reflections. Pointing to subtle differences is beyond the scope of this review. It is more interesting to note that the authors present three examples with detailed computations. They are for:

(i) the group of reflections of the tetrahedron acting on the domain and the reflection group of the equilateral triangle acting on the target;

(ii) the group of reflections of the cube acting on the domain and the reflection group of the tetrahedron acting on the target;

(iii) a crystalline structure in $\mathbb{R}^3$.

We must stress that besides the algebraic tools the authors depend on the calculus of variations and the parabolic maximum principle.

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In [S5] the authors study a generalized Painlevé equation depending on two variables,

$$\Delta y - x_1 y - 2y^3 = 0, \qquad x = (x_1, x_2) \in \mathbb{R}^2. \tag{6}$$

When we write (6) as $\Delta y = H_y(x_1, y)$, where $H(x_1, y) = \frac{1}{2}x_1 y^2 + \frac{1}{2}y^4$, then we see that (6) generalizes the phase transition model (5). Namely, the nontrivial solutions to (6) connect two branches of minima $\pm\sqrt{(-x_1)^+/2}$ of H, while nontrivial solutions to (5) connect two minima of W, in case $W(u) = \frac{1}{4}(u^2-1)^2$. Of course we expect rich behavior of solutions to (6).

The main result of this paper tells us, roughly speaking, that there is u, a solution to (6), which is a minimizer of

$$E_{P_{II}}(u) = \int_\Omega \left(\frac{1}{s}|\nabla u|^2 + \frac{1}{2}x_1 u^2 + \frac{1}{2}u^4 \right) dx$$

with respect to compactly supported perturbations. What is more important, this solution connects different types of asymptotic behavior:

(1) for $x_1 \to \infty$ solution y behaves like the Airy function (uniformly in x_2);

(2) for fixed x_1 y goes to $\pm h(x_1)$, locally in C^2 when $x_2 \to \pm\infty$, where h is the so-called Hastings-McLeod solution of (4).

The authors address also a version of (6) with a small parameter,

$$\epsilon\Delta u + \mu(x)u - u^3 = 0, \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

when μ is radial, decreasing along the rays, with a unique zero. The authors show that a solution which goes to zero as $|x| \to \infty$, after proper scaling and shifting converges locally in C^2 in the half-planes $[s_0, \infty) \times \mathbb{R}$ to a solution of (6).

In [S6], whose sole author is dr Smyrnelis, the vector valued version of the second Painlevé equation is studied,

$$\Delta y - x_1 y - 2|y|^2 y = 0, \tag{7}$$

where $y : \mathbb{R}^n \to \mathbb{R}^m$.

The motivation to study this problem stems from the observation (proved elsewhere) that a detailed description of the light intensity in a liquid crystal illuminated by a laser beam can be obtained with the help of solutions to (7).

According to the author, paper [S6] is the first attempt to construct nontrivial solutions to (7). More precisely, the goal of the paper is to construct a counterpart of the so-called standard vortex solution η for the Ginzburg-Landau system,

$$\Delta u = |u|^2 u - u,$$

where $u : \mathbb{R}^n \to \mathbb{R}^n$. This means that η is invariant with respect to the group $O(n)$, i.e. $\eta(x) = \eta_{rad}(|x|)\frac{x}{|x|}$, and η_{rad} is increasing and converging to 1 at infinity.

A precise statement of the main result in [S6] is more complicated due to the fact that we have a family of spheres in $\mathbb{R}^{n-1}$ as minimizers of the inhomogeneous potential $H(x_1, y) = \frac{1}{2}x_1|y|^2 + \frac{1}{2}|y|^4$. The choice of dimension $m = n - 1$ is dictated by the desire that for a fixed $x_1 = c$ the potential attains its minimum on a sphere in hyperplane $x_1 = c$.

This paper closing the series is definitely very technical. It shows the technical mastery of its sole author.

The papers of the habilitation achievement form a consistent series of articles of increasing difficulty. Actually, I'd rather say that there are two maximal points [S4] and [S6], if 'the increasing difficulty' is

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a partial ordering. In my opinion all the papers in this collection are fine pieces of analysis, written in different teams of authors and published in good or very good journals. In addition [S1] is written jointly with dr Smyrnelis student. This and the fact that one of the ‘maximal points’, [S6], is written solely supports the claim that dr Smyrnelis is capable of taking leadership in research.

Let me also comment on other papers by dr Smyrnelis. After graduation with his PhD degree he published 13 journal articles and a book. They cover a number of topics. Let me first comment on the book, which is devoted mainly to many aspects of the Allen-Cahn equation. It not only presents the state of the art but also offers a gentle introduction into the subject.

The topics of these papers cover a range of aspects of elliptic problems. Let me just mention some of them to stress the breadth of the scope of dr Smyrnelis’ interests. We have papers on symmetry of solutions and symmetry breaking. There are results on gradient estimates and the maximum principle. The scope of work includes Allen-Cahn eq./systems, extended Fisher-Kolmogorov eq. and there are articles on systems like Ginzburg-Landau problem.

Let me comment on [S15], which develops a clever idea to construct solutions connecting different heteroclinic orbits. Namely, one of the variables is treated like time and the author looks for $U(\cdot)$, critical points of an action functional, which for each value of t take values in a Hilbert space. This approach was subsequently applied e.g. in [S16] to find double layer solutions to the extended Fisher-Kolmogorov equation.

It is quite important to mention that the work described above was performed at very good research centers around the world, where dr Smyrnelis worked as a post-doc fellow. Dr Smyrnelis presented his results at a number of international and national conferences and workshops. Appreciation of the value of his results comes from the grant agencies. Let me emphasize that he is a recipient of the Marie Skłodowska-Curie individual fellowship. Dr Smyrnelis was also the principal investigator in a grant awarded by the Fondecyt, Chile. He was also an investigator in three more grants.

Summarizing, I would write that dr Smyrnelis is a well established researcher with a well-defined agenda of important topics in elliptic PDEs stemming from applications. His achievement has been already recognized by the scientific community. From my personal experience I can attest that dr Smyrnelis is also a good expositor. I enjoyed his presentations.

Taking into account all aspects of the evaluation I can write that dr Smyrnelis satisfies the requirements imposed by law and the customs on the candidates for habilitation. In fact I am so impressed by his work, in particular [S4] and [S6], that I would suggest granting a distinction to the habilitation.

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