Université Bourgogne Franche-Comté - Besançon
C.N.R.S. – UMR 6623
LABORATOIRE DE MATHÉMATIQUES
DE BESANÇON
16, Route de Gray - F-25030 BESANÇON CEDEX
Tél: 03 81 66 63 76
Email: gilles.lancien@univ-fcomte.fr

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Report on the habilitation thesis of Dr. Tomasz Kochanek, entitled

"The three-space problem and transfinite methods in the geometry of Banach spaces".

The thesis under review focuses on some of the results obtained by Tomasz Kochanek after his PhD thesis. As indicated by the title, the common ground of these works is the three-space problem and the use of transfinite methods, which are applied in particular to the study of the asymptotic geometry of Banach spaces and ideals of operators. One of the key tools is the so-called Szlenk index of a Banach space. This memoir is built around seven important papers by T. Kochanek and very few coauthors ([7], [5], [4], [6], [2], [3] and [1]). Let us now recall that a property (P) of Banach spaces is a three-space property (in short 3SP) if a Banach space X has property (P) whenever there exists a subspace Y of X such that Y and X/Y have property (P). At the heart of the three-space problem is the notion of twisted sums of Banach spaces. A twisted sum of Banach spaces Y and X is any exact sequence of the form

$$0 \to Y \to Z \to X \to 0,$$

which means that there is a subspace Y of Z such that Z/Y is isomorphic to X. There is a natural equivalence relation on short exact sequences. Then $\mathrm{Ext}(X, Y)$ denotes the family of equivalence classes of twisted sums of these spaces. In particular $\mathrm{Ext}(X, Y) = 0$ if every twisted sum of of Y and X is a direct sum.

The paper [7] is inspired by the fundamental work of Kalton and Peck, who described the deep links between the structure of exact sequences and the properties of quasi-linear maps. This motivates the definition, introduced by the author, of the SVM property (stability of vector measures) for Banach spaces. A Banach space X has the SVM property if there exists a finite constant $v(X)$ such that for any set algebra $\mathcal{F}$, any $\nu : \mathcal{F} \to X$ satisfying $\|\nu(A \cup B) - \nu(A) - \nu(B)\| \leq 1$ for all $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, is uniformly approximable, up to $v(X)$, by a finitely additive measure μ. One fundamental result of Kalton and Roberts is that $\mathbb{R}$ has the SVM property. A natural notion of κ-SVM property is also introduced, when the above definition is restricted to set algebras of cardinality smaller than κ. It is impossible to describe all the important results obtained by the author in [7]. Let us just mention the following ones. The κ-SVM and therefore the SVM properties are 3SP properties. If a Banach space X is complemented in its bidual, then X has SVM iff $\mathrm{Ext}(X^*, \ell_1) = 0$ iff $\mathrm{Ext}(\ell_\infty, X^{**}) = 0$ iff $\mathrm{Ext}(c_0, X) = 0$.

We now turn to the paper [5], written in collaboration with T. Kania, on the ideal of weakly compactly generated operators acting on a Banach space, which is a natural closed operator ideal that is surjective, non injective and stable by taking the adjoint or preadjoint. A very detailed study of the WCG operators acting on the long James space $J_p(\omega_1)$ is carried by the authors. In particular they show that $WCG(J_p(\omega_1))$ is the unique maximal ideal of $B(J_p(\omega_1))$. This work is extremely precise and involved. It contains an in depth study of operators on $J_p(\omega_1)$ and yields a surprising application: Every homomorphism from $B(J_p(\omega_1))$ is automatically continuous. They also obtain interesting results on the WCG operators on $C(K)$ spaces.

These last results on WCG operators on $C(K)$-spaces somewhat motivated the next paper [4] (in collaboration with Hart and Kania). In this work, the authors introduce a chain condition (C), defined for operators acting on $C(K)$-spaces, which is intermediate between being weakly compact and being WCG and, as they explain, motivated by Pełczyńki's characterization of weakly compact operators on $C(K)$-spaces. The first result of the paper is that the condition (C) is equivalent to weak compactness when K is totally disconnected. The proof is based on a very clever modification of a lemma due to Rosenthal. They give examples of WCG operators on $C(K)$ spaces failing (C) and of compact spaces K such that the identity on $C(K)$ has (C). Finally, they show that the set of all operators on $C(K)$ satisfying (C) is a closed left-sided ideal in $B(C(K))$. This last result relies on a subtle Ramsey type of result for maps defined on 2-subsets of ω_1.

The next paper [6], written with T. Kania, is a very interesting and surprising interlude. In 1975, Kottman proved that the unit sphere of an infinite dimensional normed space contains a 1^+-separated sequence. This was improved in 1981 by Elton and Odell, who showed that one can always find in the unit sphere an infinite dimensional normed space a $(1+\varepsilon)$-separated sequence for some $\varepsilon > 0$. It is then natural to ask whether one can find such separated families of higher cardinality in non separable Banach spaces. I have to say that I was very happy to discover these results while reading this thesis. Let me mention two of them. Any quasi-reflexive Banach space contains an uncountable 1^+-separated set in its unit sphere. Any unit sphere of a super-reflexive Banach space contains a $(1+\varepsilon)$-separated set of cardinality κ, for κ regular cardinal not greater than the density character of the space. I was also impressed by the elegance of the tools, such as the use of "generalized combinations" of linear functionals in the quasi-reflexive case.

We now turn to the study of the Szlenk index in relation with natural operations on Banach spaces (papers [2] and [3]). The Szlenk index was introduced by W. Szlenk in 1968 as a tool to prove the non existence of a universal separable reflexive Banach space. The definition commonly used (which coincides with the original one, for separable Banach spaces not containing ℓ_1) is the following. Let X be a Banach space, K a weak*-compact subset of its dual X^* and $\varepsilon > 0$. Then we define

$$s^1_\varepsilon(K) = \{x^* \in K, \text{ for any weak}^* - \text{neighborhood } U \text{ of } x^*,\ diam(K \cap U) \geq \varepsilon\}$$

and inductively the sets $s^\alpha_\varepsilon(K)$ for α ordinal as follows: $s^{\alpha+1}_\varepsilon(K) = s^1_\varepsilon(s^\alpha_\varepsilon(K))$ and $s^\alpha_\varepsilon(K) = \bigcap_{\beta<\alpha} s^\beta_\varepsilon(K)$ if α is a limit ordinal.
Then $Sz(K,\varepsilon) = \inf\{\alpha,\ s^\alpha_\varepsilon(K) = \emptyset\}$ if it exists and we denote $Sz(K,\varepsilon) = \infty$ otherwise. Next we define $Sz(K) = \sup_{\varepsilon>0} Sz(K,\varepsilon)$. The closed unit ball of X^* is denoted B_{X^*} and the Szlenk index of X is $Sz(X) = Sz(B_{X^*})$.
This index turned out to have many applications in linear and non linear geometry of Banach spaces. Let us now concentrate on the condition $Sz(X) \leq \omega$. A fundamental result of Knaust, Odell and Schlumprecht insures that this condition is equivalent to the existence of an equivalent asymptotically uniformly smooth norm on X. It is easy to see that the function $\varepsilon \mapsto Sz(X,\varepsilon)$ is submultiplicative. It follows that the condition $Sz(X) \leq \omega$ immediately yields the existence of a power type upper bound: $Sz(X,\varepsilon) \leq \frac{C}{\varepsilon^p}$. Let us denote $p(X)$ the infimum of all those p's. The best possible power type for a Szlenk index of an infinite dimensional Banach space is clearly 1. There is an even stronger condition than having a Szlenk power type 1 which is having a summable Szlenk index (see the memoir for the definition). The main result of [2] is that a c_0-sum of Banach spaces with uniformly summable Szlenk index has summable Szlenk index. A non trivial important example of a space with summable Szlenk index is T^*, the original Tsirelson space. The authors showed the interesting fact that $T^*(c_0)$ fails to have a summable Szlenk index (contrary to $c_0(T^*)$). In the paper [3] the authors obtain a stability result of the

Szlenk power type $p(X)$ under some general infinite sums of Banach spaces (including of course ℓ_q-sums). They also show the following optimal result: the Szlenk power type of the injective tensor product of two Banach spaces is the maximum of their Szlenk power types. These papers are based on some in depth studies of weakly null countably branching trees in a Banach space X or dually of weak* null trees in X^*.

It is now time to come back to the central three space problem. It was already known that the property $Sz(X) \leq \omega$ is a 3SP property. It was also known that if Y is a subspace of X, then $p(X) \leq p(Y) + p(X/Y)$. One of the achievements of the paper [1], written with Causey and Draga, is to prove the optimal result: $p(X)$ is the maximum of $p(Y)$ and $p(X/Y)$. In other words, having a Szlenk power type p si a 3SP. Using previously known renorming results this yields as a corollary that the supremum of the power types of AUS renormings is also a "three space invariant". However, this very deep paper contains much more than that. The authors give a thorough presentation of the asymptotic analogues of local notions such as type and cotype of Banach spaces. Very roughly speaking a local notion is a property of the finite dimensional subspaces of a Banach space, whereas an asymptotic notion is about its finite codimensional subspaces. They introduce block structures and the notions of block (basic) (sub)type/(sub)cotype of certain abstract sets $\mathcal{E}$ of pairs of semi-norms. They also define what they mean by a block finitely representable operator A on $\mathcal{E}$, which can be specialized to an identity operator to define a block finitely representable basic sequence on $\mathcal{E}$. They describe the precise links between block (basic) (sub)type/(sub)cotype of $\mathcal{E}$ and the properties of block finitely representable basic sequences on $\mathcal{E}$. The n^{th} asymptotic structure $\{A\}_n$ of an operator $A : X \to Y$ is defined in terms of properties of weakly-null trees in X and of their images by A. Then $\mathcal{A}(A) = \cup_{n \in \mathbb{N}} \{A\}_n$ is the prototype of the abstract sets $\mathcal{E}$ introduced before. The block (basic) (sub)type/(sub)cotype of $\mathcal{A}(A)$ are called block (basic) (sub)type/(sub)cotype of A (of the Banach space X if A is the identity on X). The authors also consider weak*-analogues of these. On the other hand one can also define, using the asymptotic structure of A, the crudely asymptotically finite representability of ℓ_p in A. Then, the authors exploit the definition of $\mathcal{A}(A)$, in order to describe the block (basic) (sub)type/(sub)cotype of A in terms of properties of weakly-null trees. An example of an important application is that the set of p's such that the Banach space X has asymptotic block basic type p determines the set of p's such that X admits an equivalent asymptotically uniformly smooth norm of power type p. For a Banach space X, define $s(X)$ to be the supremum of all s such that X admits an equivalent asymptotically uniformly smooth norm of power type s. This is known to be equal to the conjugate of $p(X)$ (Godefroy-Kalton-Lancien). In the last section, the authors finally show that $p(X) = \max\{p(Y), p(X/Y)\}$ or equivalently $s(X) = \min\{s(Y), s(X/Y)\}$. This proof is based on a fine study of the "three-space behavior" of $\alpha_{p,n}(X)$ and previous results of the paper. This presentation goes far beyond the applications developed by the authors that concern operator ideals or the study of a particular three-space property and provides a very general abstract setting for the study of the asymptotic structure of an operator or of a Banach space and for the extension of most of the important results on type and cotype. It should serve as a reference for future work in this direction.

It is important to mention at this point that the seven papers that I tried to describe and that are covered by this thesis represent only one aspect of the mathematical work achieved by Tomasz Kochanek after his PhD. His spectrum of research is actually very wide and diverse. I would like to mention here his result with Eva Pernecká on Lipschitz free spaces [8] and to give a brief description of it. The Lipschitz free space over a metric space M is the natural predual of the space of real valued Lipschitz functions defined on M. It has the fundamental property that any Lipschitz function between metric spaces is the trace of a bounded operator between the free spaces. This make it very important to study the linear properties of the free spaces. These questions are usually very difficult. In particular, determining when a free space si weakly sequentially complete is an important question. The main result from [8] is a progress in that direction and states that the free space over a compact subset of a super-reflexive Banach space has Pełczyński's property (V^*) and therefore is weakly sequentially complete.

Tomasz Kochanek has presented in this memoir an important and coherent set of very interesting results. He has developed various new ways to attack difficult questions. The proofs, although often very involved, are natural and elegant and allow the author to obtain many important new results in directions where the state of knowledge was already quite advanced. Each of the papers described in this report contains significant advances. I also want to underline that his research work is broader than what he has focused on in this memoir. It is indeed worth mentioning that he also has significant results on other subjects such as Lipschitz free spaces. The text of the memoir is very well written: clear, precise and organized. I think that Tomasz Kochanek is a very talented young researcher, working with success at the international level. He has obtained many important results in a quite short amount of time. In conclusion, I believe that the work presented in this memoir is of high quality and reaches the standards of a very good habilitation thesis.

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Gilles Lancien

Professor at the Laboratoire de Mathématiques de Besançon
Université Bourgogne Franche-Comté