Projective varieties and their intersection theory

Organizing Committee:
Justyna Szpond, jszpond@impan.pl
Tomasz Szemberg, tomasz.szemberg@gmail.com

Time and length of the event:
June 6 – June 18, Banach Center, Warsaw

Description:
Our project is motivated by very interesting recent developments on unexpected hypersurfaces in [2] and sets of points which project to complete intersections [1].

Definition 1 (Unexpected hypersurface)**.** *Let $V \subset \mathbb{P}^N$ be a reduced, non-degenerate subscheme of codimension at least* 2*. We say that V* admits an unexpected hypersurface *of degree d and multiplicity m, if for a general point $P \in \mathbb{P}^N$, we have*

$$\dim[I_V \cap I_P^m]_d > \max\left\{0, \dim[I_V]_d - \binom{N+m-1}{N}\right\}.$$

In other words, a subscheme admits an unexpected hypersurface if the naive dimension count fails. As the point P is assumed to be general, it is very surprising that unexpected hypersurfaces exist even in the simplest case, when V is a set of reduced points.

Harbourne, Migliore, Nagel and Teitler studied in [6] unexpected hypersurfaces admitted by certain root systems. Among others, they observed that the root system F_4, consisting of 24 points in $\mathbb{P}^3$ admits an unexpected surface of degree 4 and multiplicity 4. This surface is thus a cone over a plane quartic curve and for P (the vertex of the cone) general the points in F_4 project onto 24 mutually distinct points on this curve. Chiantini and Migliore observed in [1] that surprisingly the projection of F_4 is a complete intersection, of the quartic curve and another curve of degree 6. This led them to study the geometry of the root system F_4. Pokora, Szemberg and Szpond found in [7] a subset of 60 points in $\mathbb{P}^3$ with similar property: its projection from a general point in $\mathbb{P}^3$ is a complete intersection of curves of degree 6 and 10. On the other hand the result of [1] is in contrast to the following innocuous-looking result on curves in $\mathbb{P}^3$, which was proved independently by Diaz [3] and Giuffrida [5], see [4] for a uniform statement.

Theorem 2. *Let C and D be reduced, irreducible and non-degenerate curves in $\mathbb{P}^3$ of degree c and d respectively. Then the number m of their intersection points (ignoring the multiplicities) is subject to the following restriction*

$$m \leqslant (c-1)(d-1)+1. \tag{1}$$

In particular, this results asserts that two non-degenerate and irreducible curves in $\mathbb{P}^3$ never have as many points in common in $\mathbb{P}^3$ as curves of the same degrees in $\mathbb{P}^2$, where the number of intersection points of two curves of degrees c and d is of course the Bezout number cd.

However results of [1] show that if one drops the assumption *irreducible* in Theorem 2, then there do exist non-degenerate one dimensional subschemes in $\mathbb{P}^3$ intersecting in the Bezout number of points. In fact [1] is devoted to the classification of such subschemes and the main result asserts that they are lines intersecting in (a, b)–grids.

Definition 3 (Grid)**.** *Let a and b be positive integers. A set Z of $a \cdot b$ points in $\mathbb{P}^3$ is an $(a,b)-$*grid *if there exist two sets of lines $L_1, \ldots, L_a$ and $M_1, \ldots, M_b$ such that lines in both sets are pairwise skew and each pair of lines L_i and M_j intersect in a point of Z, thus*

$$Z = \{L_i \cap M_j,\ i = 1, \ldots, a,\ j = 1, \ldots, b\}.$$

An immediate consequence of [1] is the following

Theorem 4. *All (a,b)–grids in $\mathbb{P}^3$ have the geproci property.*

A very bold problem, we want to study is the following.

Problem 5. *Characterize subschemes of $\mathbb{P}^N$ with the geproci property.*

References

[1] Chiantini, L., and Migliore, J. Sets of points which project to complete intersections, and unexpected cones. *Trans. Amer. Math. Soc. 374*, 4 (2021), 2581–2607. With an appendix by A. Bernardi, L. Chiantini, G. Denham, G. Favacchio, B. Harbourne, J. Migliore, T. Szemberg and J. Szpond.

[2] Cook, D., Harbourne, B., Migliore, J., and Nagel, U. Line arrangements and configurations of points with an unexpected geometric property. *Compositio Mathematica 154*, 10 (2018), 2150–2194.

[3] Diaz, S. Space curves that intersect often. *Pacific Journal of Mathematics 123*, 2 (1986), 263–267.

[4] Eisenbud, D., Green, M., Hulek, K., and Popescu, S. Restricting linear syzygies: algebra and geometry. *Compos. Math. 141*, 6 (2005), 1460–1478.

[5] Giuffrida, S. On the intersection of two curves in $\mathbb{P}^3$. *Boll. Un. Mat. Ital. D (6) 5*, 1 (1986), 31–41 (1987).

[6] Harbourne, B., Migliore, J., Nagel, U., and Teitler, Z. Unexpected hypersurfaces and where to find them. *Michigan Mathematical Journal 70*, 2 (2021), 301–339.

[7] Pokora, P., Szemberg, T., and Szpond, J. Unexpected properties of the Klein configuration of 60 points in $\mathbb{P}^3$. *Research in Pairs 2020, OWP-2020-19* (2020, to appear in Michigan Math. J.).

List of participants:

Luca Chiantini, Department of Information Engineering and Mathematical Sciences
Łucja Farnik, Pedagogical University of Cracow
Giuseppe Favacchio, University of Palermo
Brian Harbourne, University of Nebraska-Lincoln
Juan Migliore, University of Notre Dame
Tomasz Szemberg, Pedagogical University of Cracow
Justyna Szpond, Polish Academy of Sciences