Geometry of the Moduli Space of Curves and Algebraic Manifolds

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60th Anniversary of the Institute of Mathematics
Polish Academy of Sciences
April 4, 2009

The first part of this talk is joint work with Kefeng Liu and Xiaofeng Sun. We study the geometry of the Teichmüller and the moduli spaces of curves.

In second part, which is joint work with Liu, Sun and also Andrey Todorov, we discuss Calabi-Yau manifolds and study their Teichmüller and moduli spaces.

Basics of the Teichmüller and Moduli Spaces

Fix an orientable surface Σ of genus $g \geq 2$.

Uniformization Theorem

Each Riemann surface of genus $g \geq 2$ can be viewed as a quotient of the hyperbolic plane $\mathbb{H}$ by a Fuchsian group. Thus there is a unique Poincaré metric, or the hyperbolic metric on Σ.

The group $Diff^+(\Sigma)$ of orientation preserving diffeomorphisms acts on the space $\mathcal{C}$ of all complex structures on Σ by pull-back.

Teichmüller Space

$$\mathcal{T}_g = \mathcal{C}/\mathit{Diff}_0^+(\Sigma)$$

where $\mathit{Diff}_0^+(\Sigma)$ is the set of orientation preserving diffeomorphisms which are isotopic to identity.

Moduli Space

$$\mathcal{M}_g = \mathcal{C}/\mathit{Diff}^+(\Sigma) = \mathcal{T}_g/\mathit{Mod}(\Sigma)$$

is the quotient of the Teichmüller space by the mapping class group where

$$\mathit{Mod}\,(\Sigma) = \mathit{Diff}^+(\Sigma)/\mathit{Diff}_0^+(\Sigma).$$

Dimension

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3.$$

$\mathcal{T}_g$ is a pseudoconvex domain in $\mathbb{C}^{3g-3}$: Bers' embedding theorem. $\mathcal{M}_g$ is a complex orbifold, it can be compactified to a projective orbifold by adding a normal crossing divisors D consisting of stable nodal curves, called the Deligne-Mumford compactification, or DM moduli.

Tangent and Cotangent Space

By the deformation theory of Kodaira-Spencer and the Hodge theory, for any point $X \in \mathcal{M}_g$,

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X)$ is the space of holomorphic quadratic differentials on X.

For $\mu \in HB(X)$ and $\phi \in Q(X)$, the duality between $T_X\mathcal{M}_g$ and $T^*_X\mathcal{M}_g$ is

$$[\mu : \phi] = \int_X \mu\phi.$$

Teichmüller metric is the L^1 norm and the WP metric is the L^2 norm. Alternatively, let
$\pi : \mathfrak{X} \to \mathcal{M}_g$ be the universal curve and let $\omega_{\mathfrak{X}/\mathcal{M}_g}$ be the relative dualizing sheaf. Then

$$\omega_{WP} = \pi_* \left(c_1 \left(\omega_{\mathfrak{X}/\mathcal{M}_g} \right)^2 \right).$$

Curvature

Let $\mathfrak{X}$ be the total space over the $\mathcal{M}_g$ and π be the projection. Pick $s \in \mathcal{M}_g$, let $\pi^{-1}(s) = X_s$. Let $s_1, \cdots, s_n$ be local holomorphic coordinates on $\mathcal{M}_g$ and let z be local holomorphic coordinate on X_s.
The Kodaira-Spencer map is

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\bar{z} \in HB(X_s).$$

The Weil-Petersson metric is

$$h_{i\bar{j}} = \int_{X_s} A_i \overline{A_j} \, dv$$

where $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\bar{z}$ is the volume form of the KE metric λ on X_s.

By the work of Royden, Siu and Schumacher, let

$$a_i = -\lambda^{-1}\partial_{s_i}\partial_{\overline{z}} \log \lambda.$$

Then

$$A_i = \partial_{\overline{z}} a_i.$$

Let η be a relative $(1,1)$ form on $\mathfrak{X}$. Then

$$\frac{\partial}{\partial s_i}\int_{X_s} \eta = \int_{X_s} L_{v_i}\eta$$

where

$$v_i = \frac{\partial}{\partial s_i} + a_i \frac{\partial}{\partial z}$$

is called the harmonic lift of $\frac{\partial}{\partial s_i}$. In the following, we let

$$f_{i\bar{j}} = A_i\overline{A}_j \text{ and } e_{i\bar{j}} = T(f_{i\bar{j}}).$$

Here $T = (\square + 1)^{-1}$ with $\square = -\lambda^{-1}\partial_z\partial_{\overline{z}}$, is the Green operator. The functions $f_{i\bar{j}}$ and $e_{i\bar{j}}$ will be the building blocks of the curvature formula.

Curvature Formula of the WP Metric

By the work of Wolpert, Siu and Schumacher, the curvature of the Weil-Petersson metric is

$$R_{i\bar{j}k\bar{l}} = -\int_{X_s} (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}})\, dv.$$

- The sign of the curvature of the WP metric can be seen directly.
- The precise upper bound $-\frac{1}{2\pi(g-1)}$ of the holomorphic sectional curvature and the Ricci curvature of the WP metric can be obtained by the spectrum decomposition of the operator $(\Box + 1)$.
- The curvature of the WP metric is not bounded from below. But *surprisingly* the Ricci and the perturbed Ricci metrics have bounded (negative) curvatures.
- The WP metric is incomplete.

Observation

The Ricci curvature of the Weil-Petersson metric is bounded above by a negative constant, one can use the negative Ricci curvature of the WP metric to define a new metric.
We call this metric the **Ricci metric**

$$\tau_{i\bar{j}} = -Ric(\omega_{WP})_{i\bar{j}}.$$

We proved the Ricci metric is complete, Poincaré growth, and has bounded geometry.
We perturbed the Ricci metric with a large constant multiple of the WP metric to define the **perturbed Ricci metric**

$$\omega_{\tilde{\tau}} = \omega_{\tau} + C\,\omega_{WP}.$$

We proved that the perturbed Ricci metric is complete, Poincaré growth and has bounded negative holomorphic sectional and Ricci curvatures, and bounded geometry.

Selected Applications of These Metrics

- Royden proved that

$$\text{Teichmüller metric} = \text{Kobayashi metric.}$$

  This implies that the isometry group of $\mathcal{T}_g$ is exactly the mapping class group.
- Ahlfors: the WP metric is Kähler, the holomorphic sectional curvature is negative.
- Masur-Wolpert: WP metric is incomplete.

Wolpert studied WP metric in great details and found many important applications in topology (relation to Thurston's work) and algebraic geometry (relation to Mumford's work).
McMullen proved that the moduli spaces of Riemann surfaces are Kähler hyperbolic, by using his metric ω_M which he obtained by perturbing the WP metric.

Theorem

All complete metrics on $\mathcal{T}_g$ and $\mathcal{M}_g$ are equivalent. Furthermore, the Carathe´odory metric, Kobayashi metric, Bergman metric and KE metric are equivalent on general homogeneous holomorphic regular manifolds.

Here, homogeneous holomorphic regular manifolds are those manifolds where the Bers embedding theorem holds. We proved this theorem using Schwarz lemma proved by me in 1973. Subsequently, S.K. Yeung published a weaker version of this theorem.

Theorem

The Ricci, perturbed Ricci and Kähler-Einstein metrics are complete, have (strongly) bounded geometry and Poincaré growth. The holomorphic sectional and Ricci curvatures of the perturbed Ricci metric are negatively pinched.

Together with L. Ji, we observed a consequence of the bounded geometry of these metrics:

Theorem

The Gauss-Bonnet Theorem hold on the moduli space equipped with the Ricci, perturbed Ricci or Kähler-Einstein metrics:

$$\int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{\tilde{\tau}}) = \int_{\mathcal{M}_g} c_n(\omega_{KE}) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

Here $\chi(\mathcal{M}_g)$ is the orbifold Euler characteristic of $\mathcal{M}_g$.

The proof is based on the Schoen-Yau's construction of canonical exhaustion functions for complete manifolds whose Ricci curvature has a lower bound.

The detailed curvature properties of the Kähler-Einstein metric can be used to prove the conjecture that the Teichmüller space cannot be embedded into the Euclidean space as a bounded smooth domain.

Algebraic-geometric consequences

Theorem

The log cotangent bundle $\overline{E} = T^*_{\overline{\mathcal{M}}_g}(\log D)$ *of the DM moduli of stable curves is stable with respect to its canonical polarization.*

Corollary

Orbifold Chern number inequality

$$c_1(\overline{E})^2 \leq \frac{6g-4}{3g-3} c_2(\overline{E}).$$

Asymptotic

- Deligne-Mumford Compactification: For a Riemann surface X, a point $p \in X$ is a node if there is a neighborhood of p which is isomorphic to the germ

  $$\{(u, v) \mid uv = 0, |u| < 1, |v| < 1\} \subset \mathbb{C}^2.$$

  A Riemann surface with nodes is called a nodal surface.
  A nodal Riemann surface is stable if each connected component of the surface subtracting the nodes has negative Euler characteristic. In this case, each connected component has a complete hyperbolic metric.
  The union of $\mathcal{M}_g$ and moduli of stable nodal curves of genus g is the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$, the DM moduli.
  $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is a divisor of normal crossings.

- Principle: To compute the asymptotic of the Ricci metric and its curvature, we work on surfaces near the boundary of $\mathcal{M}_g$. The geometry of these surfaces localize on the pinching collars.
- Model degeneration: Earle-Marden, Deligne-Mumford, Wolpert: Consider the variety

$$V = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\} \subset \mathbb{C}^3$$

and the projection $\Pi : V \to \Delta$ given by

$$\Pi(z, w, t) = t$$

where Δ is the unit disk.
If $t \in \Delta$ with $t \neq 0$, then the fiber $\Pi^{-1}(t) \subset V$ is an annulus (collar).
If $t = 0$, then the fiber $\Pi^{-1}(t) \subset V$ is two transverse disks $|z| < 1$ and $|w| < 1$.
This is the local model of degeneration of Riemann surfaces.

Asymptotic in pinching coordinates

Theorem

Let $(t_1, \cdots t_m, s_{m+1}, \cdots s_n)$ be the pinching coordinates. Then WP metric h has the asymptotic:
(1) $h_{i\bar{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1 + O(u_0))$ *for* $1 \le i \le m$*;*
(2) $h_{i\bar{j}} = O(\frac{u_i^3 u_j^3}{|t_i t_j|})$ *if* $1 \le i, j \le m$ *and* $i \neq j$*;*
(3) $h_{i\bar{j}} = O(1)$ *if* $m + 1 \le i, j \le n$*;*
(4) $h_{i\bar{j}} = O(\frac{u_i^3}{|t_i|})$ *if* $i \le m < j$*.*
Here $u_i = \frac{l_i}{2\pi}$, $l_i \approx -\frac{2\pi^2}{\log|t_i|}$ *and* $u_0 = \sum u_i + \sum |s_j|$*.*

Theorem

The Ricci metric τ has the asymptotic:

(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1+O(u_0))$ *if* $i \leq m$*;*

(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i+u_j)\right)$ *if* $i,j \leq m$ *and* $i \neq j$*;*

(3) $\tau_{i\bar{j}} = O\left(\frac{u_i^2}{|t_i|}\right)$ *if* $i \leq m < j$*;*

(4) $\tau_{i\bar{j}} = O(1)$ *if* $i,j \geq m+1$.

Finally we derive the curvature asymptotic:

Theorem

The holomorphic sectional curvature of the Ricci metric τ satisfies

$$\widetilde{R}_{i\bar{i}i\bar{i}} = -\frac{3u_i^4}{8\pi^4|t_i|^4}(1+O(u_0)) > 0 \quad if \quad i \leq m$$

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \quad if \quad i > m.$$

To prove that the holomorphic sectional curvature of the perturbed Ricci metric

$$\omega_{\tilde{\tau}} = \omega_{\tau} + C\,\omega_{_{WP}}$$

is negatively pinched, we notice that it remains negative in the degeneration directions when C varies and is dominated by the curvature of the Ricci metric.
When C large, the holomorphic sectional curvature of $\tilde{\tau}$ can be made negative in the interior and in the non-degeneration directions near boundary from the negativity of the holomorphic sectional curvature of the WP metric.
The estimates of the bisectional curvature and the Ricci curvature of these new metrics are long and complicated computations.
The lower bound of the injectivity radius of the Ricci and perturbed Ricci metrics and the KE metric on the Teichmüller space is obtained by using Bers embedding theorem, minimal surface theory and the boundedness of the curvature of these metrics.

Bounded Geometry of the KE Metric

The first step is to perturb the Ricci metric by using the Kähler-Ricci flow

$$\begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -(R_{i\bar{j}} + g_{i\bar{j}}) \\ g(0) = \tau \end{cases}$$

to avoid complicated computations of the covariant derivatives of the curvature of the Ricci metric.

For $t > 0$ small, let $h = g(t)$ and let g be the KE metric. We have

- h is equivalent to the initial metric τ and thus is equivalent to the KE metric.
- The curvature and its covariant derivatives of h are bounded.

Then we consider the Monge-Amperé equation

$$\log\det(h_{i\bar{j}} + u_{i\bar{j}}) - \log\det(h_{i\bar{j}}) = u + F$$

where $\partial\overline{\partial}u = \omega_g - \omega_h$ and $\partial\overline{\partial}F = Ric(h) + \omega_h$.

- Equivalences: $\partial\overline{\partial}u$ has C^0 bound.
- The strong bounded geometry of h implies $\partial\overline{\partial}F$ has C^k bounds for $k \geq 0$.

Stability of the Log Cotangent Bundle $\overline{E}$

The proof of the stability needs the detailed understanding of the boundary behaviors of the KE metric to control the convergence of the integrals of the degrees.

- As a current, $\omega_{_{KE}}$ is closed and represent the first Chern class of $\overline{E}$.
$$[\omega_{_{KE}}] = c_1(\overline{E}).$$
- The singular metric $g^*_{_{KE}}$ on $\overline{E}$ induced by the KE metric defines the degree of $\overline{E}$.
$$\deg(\overline{E}) = \int_{\mathcal{M}_g} \omega^n_{_{KE}}.$$

- The degree of any proper holomorphic sub-bundle F of $\overline{E}$ can be defined using $g^*_{KE}\mid_F$.

$$\deg(F) = \int_{\mathcal{M}_g} -\partial\overline{\partial} \log \det \left(g^*_{KE} \mid_F \right) \wedge \omega^{n-1}_{KE}.$$

Also needed is a basic non-splitting property of the mapping class group and its subgroups of finite index.

Goodness and Negativity

Now I will discuss the goodness of the Weil-Petersson metric, the Ricci and the perturbed Ricci metrics in the sense of Mumford, and their applications in understanding the geometry of moduli spaces.

The question that WP metric is good or not has been open for many years, according to Wolpert. Corollaries include:

- Chern classes can be defined on the moduli spaces by using the Chern forms of the WP metric, the Ricci or the perturbed Ricci metrics; the L^2-index theory and fixed point formulas can be applied on the Teichmüller spaces.
- The log cotangent bundle is Nakano positive; vanishing theorems of good cohomology; rigidity of the moduli spaces.

Goodness of Hermitian Metrics

For an Hermitian holomorphic vector bundle (F, g) over a closed complex manifold M, the Chern forms of g represent the Chern classes of F. However, this is no longer true if M is not closed since g may be singular.

- X: quasi-projective variety of $\dim_{\mathbb{C}} X = k$ by removing a divisor D of normal crossings from a closed smooth projective variety $\overline{X}$.
- $\overline{E}$: a holomorphic vector bundle of rank n over $\overline{X}$ and $E = \overline{E}\,|_X$.
- h: Hermitian metric on E which may be singular near D.

Mumford introduced conditions on the growth of h, its first and second derivatives near D such that the Chern forms of h, as currents, represent the Chern classes of $\overline{E}$.
We cover a neighborhood of $D \subset \overline{X}$ by finitely many polydiscs

$$\left\{ U_\alpha = \left(\Delta^k, (z_1, \cdots, z_k) \right) \right\}_{\alpha \in A}$$

such that $V_\alpha = U_\alpha \setminus D = (\Delta^*)^m \times \Delta^{k-m}$. Namely, $U_\alpha \cap D = \{z_1 \cdots z_m = 0\}$. We let $U = \bigcup_{\alpha \in A} U_\alpha$ and $V = \bigcup_{\alpha \in A} V_\alpha$. On each V_α we have the local Poincaré metric

$$\omega_{P,\alpha} = \frac{\sqrt{-1}}{2} \left(\sum_{i=1}^{m} \frac{1}{2|z_i|^2 \left(\log |z_i|\right)^2} dz_i \wedge d\overline{z}_i + \sum_{i=m+1}^{k} dz_i \wedge d\overline{z}_i \right).$$

Definition

Let η be a smooth local p-form defined on V_α.

- We say η has Poincaré growth if there is a constant $C_\alpha > 0$ depending on η such that

$$|\eta(t_1, \cdots, t_p)|^2 \leq C_\alpha \prod_{i=1}^{p} \|t_i\|^2_{\omega_{P,\alpha}}$$

for any point $z \in V_\alpha$ and $t_1, \cdots, t_p \in T_z X$.

- η is good if both η and $d\eta$ have Poincaré growth.

Definition

An Hermitian metric h on E is good if for all $z \in V$, assuming $z \in V_\alpha$, and for all basis $(e_1, \cdots, e_n)$ of $\overline{E}$ over U_α, if we let $h_{i\bar{j}} = h(e_i, e_j)$, then

- $\left|h_{i\bar{j}}\right|, (\det h)^{-1} \leq C \left(\sum_{i=1}^m \log |z_i|\right)^{2n}$ for some $C > 0$.
- The local 1-forms $\left(\partial h \cdot h^{-1}\right)_{\alpha\gamma}$ are good on V_α. Namely the local connection and curvature forms of h have Poincaré growth.

Properties of Good Metrics

- The definition of Poincaré growth is independent of the choice of U_α or local coordinates on it.
- A form $\eta \in A^p(X)$ with Poincaré growth defines a p-current $[\eta]$ on $\overline{X}$. In fact we have

$$\int_X |\eta \wedge \xi| < \infty$$

  for any $\xi \in A^{k-p}(\overline{X})$.
- If both $\eta \in A^p(X)$ and $\xi \in A^q(X)$ have Poincaré growth, then $\eta \wedge \xi$ has Poincaré growth.
- For a good form $\eta \in A^p(X)$, we have $d[\eta] = [d\eta]$.

The importance of a good metric on E is that we can compute the Chern classes of $\overline{E}$ via the Chern forms of h as currents.

Mumford has proved:

Theorem

Given an Hermitian metric h on E, there is at most one extension $\overline{E}$ of E to $\overline{X}$ such that h is good.

Theorem

If h is a good metric on E, the Chern forms $c_i(E, h)$ are good forms. Furthermore, as currents, they represent the corresponding Chern classes $c_i(\overline{E}) \in H^{2i}(\overline{X}, \mathbb{C})$.

With the growth assumptions on the metric and its derivatives, we can integrate by part, so Chern-Weil theory still holds.

Good Metrics on Moduli Spaces

Now we consider the metrics induced by the Weil-Petersson metric, the Ricci and perturbed Ricci metrics on the logarithmic extension of the holomorphic tangent bundle over the moduli space of Riemann surfaces.

Our theorems hold for the moduli space of Riemann surfaces with punctures.

Let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces with $g \geq 2$ and let $\overline{\mathcal{M}}_g$ be its Deligne-Mumford compactification. Let $n = 3g - 3$ be the dimension of $\mathcal{M}_g$ and let $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ be the compactification divisor.

Let $\overline{E} = T^*_{\overline{\mathcal{M}}_g}(\log D)$ be the logarithmic cotangent bundle over $\overline{\mathcal{M}}_g$.

For any Kähler metric p on $\mathcal{M}_g$, let p^* be the induced metric on $\overline{E}$. We know that near the boundary $\{t_1 \cdots t_m = 0\}$,

$$\left(\frac{dt_1}{t_1}, \cdots, \frac{dt_m}{t_m}, dt_{m+1}, \cdots, dt_n\right)$$

is a local holomorphic frame of $\overline{E}$.

In these notations, near the boundary the log tangent bundle $F = T_{\overline{\mathcal{M}}_g}(-\log D)$ has local frame

$$\left\{t_1 \frac{\partial}{\partial t_1}, \cdots, t_m \frac{\partial}{\partial t_m}, \frac{\partial}{\partial t_{m+1}}, \cdots, \frac{\partial}{\partial t_n}\right\}.$$

We have proved several results about the goodness of the metrics on moduli spaces. By very subtle analysis on the metric, connection and curvature tensors.

We first proved the following theorem:

Theorem

The metric h^* on the logarithmic cotangent bundle $\overline{E}$ over the DM moduli space induced by the Weil-Petersson metric is good in the sense of Mumford.

Based on the curvature formulae of the Ricci and perturbed Ricci metrics we derived before, we have proved the following theorem from much more detailed and harder analysis: estimates over 80 terms.

Theorem

The metrics on the log tangent bundle $T_{\overline{\mathcal{M}}_g}(-\log D)$ over the DM moduli space induced by the Ricci and perturbed Ricci metrics are good in the sense of Mumford.

A direct corollary is

Theorem

The Chern classes $c_k\left(T_{\overline{\mathcal{M}}_g}(-\log D)\right)$ *are represented by the Chern forms of the Weil-Petersson, Ricci and perturbed Ricci metrics.*

This in particular means we can use the explicit formulas of Chern forms of the Weil-Petersson metric derived by Wolpert to represent the classes, as well as those Chern forms of the Ricci and the perturbed Ricci metrics.

Another important corollary of the goodness of these metrics is the Gauss-Bonnet theorem of the WP metric:

Theorem

The Gauss-Bonnet Theorem hold on the moduli space equipped with the Weil-Petersson metric:

$$\int_{\mathcal{M}_g} c_n(\omega_{WP}) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

The proof depends on the goodness of the WP and Ricci metrics which give control of the Bott-Chern forms between them.

Dual Nakano Negativity of WP Metric

It was shown by Ahlfors, Royden and Wolpert that the Weil-Petersson metric have negative Riemannian sectional curvature.

Schumacher showed that the curvature of the WP metric is strongly negative in the sense of Siu.

In 2005, we showed that the curvature of the WP metric is dual Nakano negative.

Applications

As corollaries of goodness and the positivity or negativity of the metrics, first we directly obtain:

Theorem

The Chern numbers of the log cotangent bundle of the moduli spaces of Riemann surfaces are positive.

We have several corollaries about cohomology groups of the moduli spaces. More generally, we setup the theory of good cohomology for general quasi-projective manifolds equipped with good metrics.

Let $\overline{X}$ be a projective manifold (orbifold) of dimension $\dim_{\mathbb{C}} \overline{X} = n$ and let $D \subset \overline{X}$ be a divisor of normal crossings. Let $X = \overline{X} \setminus D$. Let ω_g be a good Kähler metric on X with respect to the log extension of the holomorphic tangent bundle.

An important and direct application of the goodness of the WP metric and its dual Nakano negativity together with the goodness of the Ricci metrics is the vanishing theorem of the good cohomology group:

Theorem

The good cohomology groups

$$H_G^{0,q}\left((\mathcal{M}_g,\omega_\tau),\left(T_{\overline{\mathcal{M}}_g}(-\log D),\omega_{WP}\right)\right)=0$$

unless $q=n$.

We put the Ricci metric on the base manifold to avoid the incompleteness of the WP metric.

To prove this, we first consider the Kodaira-Nakano identity

$$\Box_{\overline{\partial}} = \Box_{\nabla} + \sqrt{-1}\left[\nabla^2, \Lambda\right].$$

We then apply the dual Nakano negativity of the WP metric to get the vanishing theorem by using the goodness to deal with integration by part. There is no boundary term.

Remark

- As corollaries, we also have: the moduli space of Riemann surfaces is rigid: no infinitesimal holomorphic deformation.
- We are proving that the KE and Bergman metric are also good metrics and other applications to algebraic geometry and topology.

Calabi-Yau Teichmüller Spaces

The Calabi-Yau (CY) manifolds and their Teichmüller and moduli spaces are central objects in many subjects of mathematics and string theory. A CY n-fold is a complex manifold X of dimension n such that the canonical bundle K_X is trivial and the Hodge numbers $h^{p,0}(X) = 0$ for $0 < p < n$.

We fix a CY manifold X, an ample line bundle L over X and a basis B of $H_n(X, \mathbb{Z})/Tor$ (marking). Let $\mathcal{T} = \mathcal{T}_L(X)$ be the Teichmüller space of X with respect to the polarization L which leaves the marking B invariant. We let $\mathcal{M} = \mathcal{M}_L(X)$ be the moduli space.

Metrics on the Teichmüller and Moduli Spaces

There are two natural and important metrics on $\mathcal{T}$ and $\mathcal{M}$: the Weil-Petersson metric which is the induced L^2 metric, and the Hodge metric which is the pull-back of the Griffiths-Schmid metric on the classifying space by the period map.

For any point $p \in \mathcal{T}$, let X_p be the corresponding CY manifold. By the Kodaira-Spencer theory, we have the identification $T_p^{1,0}\mathcal{T} \cong H^1(X_p, T_{X_p}^{1,0}) \cong H^{0,1}(X_p, T_{X_p}^{1,0})$ where the last identification is with respect to the CY metric. For any harmonic Beltrami differentials $\phi, \psi \in H^{0,1}(X_p, T_{X_p}^{1,0})$ the WP metric is the L^2 metric

$$(\phi, \psi)_{WP} = \int_{X_p} \phi_{\bar{j}}^{i} \overline{\psi_{\bar{k}}^{l}} g_{i\bar{l}} g^{k\bar{j}} \, dV_g$$

where ω_g is the CY metric and dV_g is its volume form.

There are explicit formulae for the curvature of the WP metric. In the case $n = 3$, the curvature is due to Strominger by using Yukawa couplings.

Flat Coordinates

By the Kodaira-Spencer theory, if $\pi : \mathfrak{X} \to B$ is a holomorphic map between complex manifolds such that π_* has full rank everywhere and the fibers are connected, if $0 \in B$ then every vector $v \in T_0^{1,0}B$ can be identified with a Beltrami differential $\phi \in \check{H}^1(X_0, T_{X_0}^{1,0})$.

In general, if $M_0 = (X, J_0)$ is a complex manifold and $\phi \in A^{0,1}(M_0, T_{M_0}^{1,0})$ is a Beltrami differential with small norm, we can define a new almost complex structure J_ϕ by requiring $\Omega_\phi^{1,0} = (I + \phi)(\Omega_0^{1,0})$ and $\Omega_\phi^{0,1} = (I + \overline{\phi})(\Omega_0^{0,1})$.

The almost complex structure J_ϕ is integrable and $M_\phi = (X, J_\phi)$ is a complex manifold if

$$\overline{\partial}\phi = \frac{1}{2}[\phi, \phi]$$

which implies $\mathbb{H}([\phi, \phi]) = 0$ (Kuranishi map).

Fix a Kähler manifold M_0 and let $\eta_1, \cdots, \eta_m$ be an orthonormal basis of $H^{0,1}(M_0, T^{1,0}_{M_0})$. The Kuranishi equation is

$$\phi(t) = \sum_{i=1}^{m} t_i \eta_i + \frac{1}{2}\overline{\partial}^* G[\phi(t), \phi(t)].$$

This equation together with $\mathbb{H}([\phi(t), \phi(t)]) = 0$ imply

$$\begin{aligned} \overline{\partial}\phi(t) =& \frac{1}{2}[\phi(t), \phi(t)] \\ \overline{\partial}^* \phi(t) =& 0 \\ \mathbb{H}(\phi(t)) =& \sum_{i=1}^{m} t_i \eta_i. \end{aligned}$$

Fix $p \in \mathcal{T}$ and let M_p be the corresponding CY n-fold. Let $N = h^{n-1,1}(M_p)$ be the dimension of the Teichmüller and the moduli spaces. Let $\phi_1, \cdots, \phi_N \in \mathbb{H}^{0,1}(M_p, T^{1,0}_{M_p})$ be a harmonic basis with respect to the polarized CY metric. Then there is a unique power series $\phi(\tau) = \sum_{i=1}^{N} \tau_i \phi_i + \sum_{|I| \geq 2} \tau^I \phi_I$ which converges for $|\tau| < \epsilon$ such that

$$\overline{\partial}\phi(\tau) = \frac{1}{2}[\phi(\tau), \phi(\tau)]$$

$$\overline{\partial}^* \phi(\tau) = 0$$

$$\phi_I \lrcorner \Omega_0 = \partial \psi_I$$

for $|I| \geq 2$ where we use the polarized CY metric and Ω_0 is a nowhere vanishing holomorphic $(n, 0)$-form on M_p. The coordinates $\tau = (\tau_1, \cdots, \tau_N)$ is the flat coordinates at p (Todorov gauge).

Canonical Family

Fix Ω_0 on M_p and pick local coordinates z on M_p such that

$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_n.$$

For $|\tau| < \epsilon$ let

$$\Omega_\tau = \bigwedge_{i=1}^{n} (dz_i + \phi(\tau)(dz_i)).$$

The family Ω_τ is a canonical holomorphic family of nowhere vanishing holomorphic $(n, 0)$-forms. In the cohomology level, it has the expansion

$$[\Omega_\tau] = [\Omega_0] + \sum_{i=1}^{N} \tau_i [\phi_i \lrcorner \Omega_0] + \Xi$$

where $\Xi \in \bigoplus_{k=2}^{n} H^{n-k,k}(M_p)$.

A powerful method to study $\mathcal{T}$ is to use the variation of Hodge structures. Let D be the classifying space of polarized Hodge structures associated to X. It is well known that D is a complex homogeneous manifold. Furthermore, $D = G_{\mathbb{R}}/K_1$ is a quotient of a real semisimple Lie group by a compact subgroup.

Let K be the maximal compact subgroup containing K_1. We have the following composition of maps

$$\mathcal{T}_L(X) \to G_{\mathbb{R}}/K_1 \to G_{\mathbb{R}}/K$$

where the first map is the period map and the second map is the natural projection. Since $G_{\mathbb{R}}/K$ is symmetric of noncompact type, by using the local Torelli theorem and the Griffiths transversality, we know that the pull-back of the Killing metric on $G_{\mathbb{R}}/K$ define a metric on $\mathcal{T}$ which is called the Hodge metric.

It was proved by Griffiths and Schmid that the Hodge metric ω_H is Kähler and its bisectional curvature is nonpositive and its holomorphic sectional curvature has negative upper bound.

Corollary

(Local Torelli) For distinct points $q_1, q_2 \in \mathcal{T}$ which are sufficiently close to $r \in \mathcal{T}$, we have

$$p(q_1) \neq p(q_2) \in G_{\mathbb{R}}/K_1.$$

Corollary

For each point $p \in \mathcal{T}$, let ω_p be the Kähler form of the unique CY metric on M_p in the polarization class $[L]$. Then ω_p is invariant. Namely,

$$\nabla^{GM}\omega_p = 0.$$

Furthermore, since $\mathcal{T}$ is simply connected, we know that ω_p is a constant section of the trivial bundle $A^2(X, \mathbb{C})$ over $\mathcal{T}$.

Global Transversality Theorem

An important observation of the period map from the Teichmüller space to the period domain is the global transversality theorem. Fix a point $p \in \mathcal{T}$, the Hodge filtration $F^n(p) \subset \cdots \subset F^0(p) = H_{\mathbb{C}}$ of M_p satisfies

$$F^k(p) \oplus \overline{F^{n-k+1}(p)} = H_{\mathbb{C}}.$$

We showed

Theorem

For any point $q \in \mathcal{T}$, one has

$$F^k(p) \oplus \overline{F^{n-k+1}(q)} = H_{\mathbb{C}}.$$

Namely, if $Pr_p^k : H_{\mathbb{C}} \to F^k(p)$ is the natural projection map. Then

$$Pr_p^k\Big|_{F^k(q)} : F^k(q) \to F^k(p)$$

is a linear isomorphism.

Global Torelli Theorem

Let $G_{\mathbb{R}}/K_1$ be the classifying space of polarized Hodge structures with data from a Calabi-Yau M. Let $p : \mathcal{T} \to G_{\mathbb{R}}/K_1$ be the period map. Then we proved the following global Torelli theorem

Theorem

The period map

$$p : \mathcal{T} \longrightarrow G_{\mathbb{R}}/K_1$$

is injective.

The proof relies on tracing the cohomology classes of the holomorphic $(n,0)$-forms by using the canonical expansion. We extend the flat coordinate lines by realizing them as geodesics of a holomorphic flat connection on $\mathcal{T}$. This connection tied to the "Frobineous" structure on $\mathcal{T}$ closely. Another important ingredient is to extend the canonical family across singularities in $\mathcal{T}$ with finite Hodge distance.

Existence of Kähler-Einstein Metric

Theorem

The extended Teichmüller space $\tilde{\mathcal{T}}$ is a domain of holomorphy. Thus there exists a unique Kähler-Einstein metric on $\tilde{\mathcal{T}}$, the Hodge metric completion of the Teichmüller space.

In order to prove this, we construct a potential function of the Hodge metric on $\mathcal{T}$ which is a nice exhaustion function. The existence of Kähler-Einstein metric follows from the general construction of S.-Y. Cheng and Yau.

Holomorphic Embedding Theorem

Based on the global transversality theorem, the global Torelli theorem and the expansion of canonical family, we can embed the Teichmüller space holomorphically into the Euclidean space of same dimension.

Theorem

For any point $p \in \mathcal{T}$, let $T_p^{1,0}\mathcal{T}$ be the holomorphic tangent space. Then there is a natural map $g_p : \mathcal{T} \to T_p^{1,0}\mathcal{T}$ which is holomorphic and injective.

For higher-dimensional algebraic varieties, there is a natural generalization of the Poincaré metric which is given by the Kähler-Einstein metric. However, the Weil-Petersson metric associated to the Kähler-Einstein metric with negative scalar curvature does not seem to have very good properties.

The Weil-Petersson metric for the Calabi-Yau manifold is more interesting. Its curvature tensor contains the important information of the Yukawa coupling tensor.

For the rest of this talk, I will describe some joint work with Chen-Yu Chi.

We look at birational geometry of an algebraic variety M. Consider the vector space

$$V = H^0(M, mK_M)$$

where $m > 0$ and K_M is the canonical line bundle. There is a natural norm defined on V given by

$$\|s\| = \int_M (s\bar{s})^{\frac{1}{m}} \ .$$

The vector space $(V, \| \;\|)$ is an invariant of the birational geometry of M.

This normed space is very rich in geometry. We demonstrate this by proving a Torelli type theorem.

Theorem

$(V, \| \;\|)$ *determines* M *birationally, at least when* mK_M *has no based point and* $m \geq C$ *. The constant* C *does not depend on* M *if* $\dim_{\mathbb{C}} M = 2$ *.*

The above theorem can be considered as a Torelli type theorem for birational geometry. We are in the process of characterizing these normed vector spaces that come from algebraic varieties, We will also give effective methods to compute such invariants.