$\delta^\sharp(2,2)$-ideal hypersurfaces of dimension 5 in centroaffine differential geometry

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Abstract

Let M^n be an n-dimensional Riemannian manifold. Given integers $n \geq 3$ and $k \geq 1$, we denote by $\mathcal{S}(n,k)$ the finite set consisting of all k-tuples $(n_1,\ldots,n_k)$ of integers satisfying $2 \leq n_1,\cdots,n_k < n$ and $n_1+\cdots+n_k \leq n$. Moreover, we denote by $\mathcal{S}(n)$ the union $\cup_{k\geq 1}\mathcal{S}(n,k)$. For each $(n_1,\ldots,n_k) \in \mathcal{S}(n)$ and each $p \in M^n$, the invariant $\delta(n_1,\ldots,n_k)(p)$ is defined by

$$\delta(n_1,\ldots,n_k)(p) = \hat{\tau}(p) - \inf\{\hat{\tau}(L_1)+\cdots+\hat{\tau}(L_k)\},$$

where $\hat{\tau}(p)$ is the scalar curvature of M^n at p, $\hat{\tau}(L_i)$ is the scalar curvature of L_i which is a subspace of T_pM^n with $\dim L_i = n_i$, $i = 1,\ldots,k$ and $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM^n, (cf. [2]). This invariant was used to determine an optimal lower bound for the mean curvature vector of the submanifolds of real space forms. Submanifolds attaining this bound are said to be ideal submanifolds.

Such a kind of invariant can be introduced in centroaffine differential geometry as follows:

$$\delta^\sharp(n_1,\ldots,n_k)(p) = \hat{\tau}(p) - \sup\{\hat{\tau}(L_1)+\cdots+\hat{\tau}(L_k)\}.$$

Theorem 0.1 *Let M^n be a definite centroaffine hypersurface of $\mathbb{R}^{n+1}$. Take $\epsilon = 1$ (respectively, $\epsilon = -1$) if M^n is positive (respectively, negative) definite. Then, for each k-tuple $(n_1,\ldots,n_k) \in \mathcal{S}(n)$ with $n_1+\ldots+n_k < n$, we have*

$$\begin{aligned}\delta^\sharp(n_1,\ldots,n_k) \quad \geq \quad & -\frac{n^2\left(n-\sum_{i=1}^k n_i + 3k - 1 - 6\sum_{i=1}^k \frac{1}{2+n_i}\right)}{2\left(n-\sum_{i=1}^k n_i + 3k + 2 - 6\sum_{i=1}^k \frac{1}{2+n_i}\right)}\|T^\sharp\|^2 \\ & +\frac{1}{2}\left(n(n-1) - \sum_{i=1}^k n_i(n_i-1)\right)\epsilon,\end{aligned}$$

where $T^\sharp$ is Tchebychev vector field. The equality case of the above inequality holds at a point $p \in M^n$ if and only if one has

- $K^A_{BC} = 0$ *if* A, B, C *are mutually different and not all in the same* Δ_i *with* $i \in \{1, \ldots, k\}$,
- $K^{\alpha_i}_{\alpha_j\alpha_j} = K^{\alpha_i}_{rr} = \sum_{\beta_i \in \Delta_i} K^{\alpha_i}_{\beta_i\beta_i} = 0$ *for* $i \neq j$,
- $K^r_{rr} = 3K^r_{ss} = (n_i + 2)K^r_{\alpha_i\alpha_i}$ *for* $r \neq s$.

Here, K *is the difference tensor. Moreover,* $\Delta_1 = (1, ..., n_1), \Delta_2 = (n_1+1, ..., n_1+n_2), ..., \Delta_k = (n_1+...+n_{k-1}+1, ..., n_1+...+n_k), \Delta_{k+1} = (n_1 + ... + n_k + 1, ..., n)$*. Furthermore,* $A, B, C \in \{1, \ldots, n\}$*,* $i, j \in \{1, \ldots, k\}$*,* $\alpha_i, \beta_i \in \Delta_i$ *and* $r, s \in \Delta_{k+1}$.

A centroaffine immersion of M^n into R^{n+1} is called $\delta^\sharp(n_1, \ldots, n_k)$-*ideal* if it satisfies the equality case of the above inequality identically. Moreover, it is called *ideal* if it is $\delta^\sharp(n_1, \ldots, n_k)$-ideal for the corresponding $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. So far, most results in this area have only been related with 3- and 4-dimensional $\delta^\sharp(2)$-ideal centroaffine hypersurfaces (See [1]-[5].).

In this talk which is based on a joint work with Luc Vrancken in [6], we classify $\delta^\sharp(2, 2)$-ideal hypersurfaces of dimension 5 in centroaffine differential geometry.

References

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