Some Cheeger-Gromov-Taylor Type Compactness Theorems for Ricci Solitons

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Abstract. An important problem in Riemannian geometry is to investigate the relation between topology and geometric structure on Riemannian manifolds. The celebrated theorem of S. B. Myers [8] guarantees the compactness of a complete Riemannian manifold under some positive lower bound on the Ricci curvature. This theorem can be considered as a topological obstruction for a complete Riemannian manifold to have a positive lower bound on the Ricci curvature. On the other hand, J. Lohkamp [6] proved that in dimension at least three, any manifold admits a complete Riemannian metric with negative Ricci curvature. Hence, in dimension at least three, there are no topological obstructions to the existence of a complete Riemannian metric with negative Ricci curvature. To give an interesting compactness criterion for complete Riemannian manifolds is one of the most important problems in Riemannian geometry, and the Myers theorem has been widely generalized in various directions by many authors.

The aim of this talk is to discuss the compactness of complete Ricci solitons. Ricci solitons were introduced by R. Hamilton in 1982 and are natural generalizations of Einstein manifolds. They correspond to self-similar solutions to the Ricci flow and often arise as singularity models of the flow. The importance of Ricci solitons was demonstrated by G. Perelman, where Ricci solitons played crucial roles in his affirmative resolution of the Poincaré conjecture. In this talk, after we reviewed basic facts on Ricci solitons, we will establish some new compactness theorems for complete shrinking Ricci solitons. Our results can be regarded as natural generalizations of the compactness theorem due to J. Cheeger, M. Gromov, and M. Taylor [1] and improve previous compactness theorems obtained by M. Fernández-López and E. García-Río [2], M. Limoncu [4, 5], Z. Qian [9], and G. Wei and W. Wylie [10].

If time permits, after making a brief review of Sasaki geometry, we will give a new compactness theorem for complete Sasaki manifolds. This result can also be regarded as a natural generalization of the compactness theorem due to J. Cheeger, M. Gromov, and M. Taylor [1] and improves Myers type theorems due to I. Hasegawa and M. Seino [3], and Y. Nitta [7].

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