“ An inclusive immersion into a quaternionic manifold and its invariants ”

by Kazuyuki HASEGAWA (Kanazawa University, Japan)

Abstract. We say that (M, Q) is a quaternionic manifold with the quaternionic structure Q if Q is a subbundle of $\mathrm{End}(TM)$ with $\mathrm{rank}Q = 3$ which is locally spanned by I_1, I_2, I_3 satisfying $I_1^2 = I_2^2 = I_3^2 = -id$, $I_1I_2 = -I_2I_1 = I_3$, and there exists a torsion free affine connection ∇ which preserves Q. Such a torsion free affine connection ∇ is called a quaternionic connection. Note that the quaternionic connection is not unique. Our interests are objects and properties which are independent of the choice of quaternionic connections. Let Σ be an oriented surface and $f : \Sigma \to M$ an immersion. If $f_{*x}(T_x\Sigma)$ is contained in a real 4-dimensional quaternionic subspace of $T_{f(x)}M$ for each point $x \in \Sigma$, then $f : \Sigma \to M$ is called an inclusive immersion or Σ is called an inclusive surface. If f is an inclusive immersion from an oriented surface, then there exists a unique lift $I_1 : \Sigma \to \mathcal{Z}$ such that $I_1(x)$ preserves $f_*(T_x\Sigma)$ at each point $x \in \Sigma$ and the induced complex structure I on Σ is compatible with the given orientation of Σ, where $\mathcal{Z}$ is the twistor space of M. See [AM]. The map I_1 is called the natural twistor lift of f.

We introduce a quaternionic invariant for an inclusive immersion into a quaternionic manifold, which will be denoted by $\mathcal{W}_Q$ in this talk. When $M = \mathbf{H}P^1(\cong S^4)$, the functional $\mathcal{W}_Q$ coincides with the conformal Willmore functional. Therefore we may consider that $\mathcal{W}_Q$ is a candidate for a quaternionic object of the Willmore functional, because a four-dimensional quaternionic manifold is simply an oriented manifold with a conformal structure. The lower bound of this invariant is given by topological invariant and the equality case can be characterized in terms of the natural twistor lift. When $M = \mathbf{H}P^n$ and the natural twistor lift $I_1 : \Sigma \to \mathbf{C}P^{2n+1}$ is holomorphic, we obtain a relation between the quaternionic invariant and the degree of the image of the natural twistor lift as an algebraic curve. Moreover the first variation formula for the invariant is obtained. As an application of the formula, if the natural twistor lift is a harmonic section, then the surface is a stationary point under any variations such that the induced complex structures do not vary. This is a quaternionic version and a generalized result in [BC] for a constrained Willmore surface.

References.

[AM] D. Alekseevsky and S. Marchiafava, A twistor construction of Kähler submanifolds of a quaternionic Kähler manifold, Ann. Mat. Pura Appl. (4) 184 (2005), 53-74.
[BC] F. Burstall and D. Calderbank, Conformal submanifold geometry I-III, arXiv:1006.5700.
[H] K. Hasegawa, An inclusive immersion into a quaternionic manifold and its invariants, to appear in Manuscripta Math.