Integrable complex structures on some 6-dimensional Lie algebras

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A complex structure on a Lie algebra $\mathfrak{h}$ is an endomorphism $\mathcal{J} : \mathfrak{h} \to \mathfrak{h}$ such that $\mathcal{J}^2 = -id$. It is said to be integrable if

$$N(v, w) = -[v, w] + [\mathcal{J}v, \mathcal{J}w] - \mathcal{J}[\mathcal{J}v, w] - \mathcal{J}[v, \mathcal{J}w] = 0$$

for all $v, w \in \mathfrak{h}$, cf. [11]. The problem of determining whether a given Lie algebra admits an integrable complex structure or not is a well established and not an easy one, cf. [11]. During the talk we would like to present the recent progress in that subject, cf. [9, 1, 3, 5]. Just for the motivation, complex structures on a Lie algebra $\mathfrak{h}$ correspond to left invariant almost complex structures on an associated Lie group H, what is more, the integrability of the first one is equivalent to the integrability of the second one. More interestingly, in the case when $\mathfrak{h}$ is nilpotent or solvable, (integrable) complex structures on $\mathfrak{h}$ correspond to left invariant (integrable) almost complex structures on an assassinated nilmanifold or solvmanifold (quotient of an associated group by a cocompact lattice, if there is such). These manifolds form a very nice class from couple of reasons. One can obtain a lot of information about them from the Lie algebra of the group they are quotients of, for example a classical result state that the de Rham cohomology of them is just the Lie algebra cohomology, cf. [6], the same is true for the Dolbeault cohomology of some nilmanifolds, cf. [4]. This is a huge simplification in a contrast with a completely abstract manifold. They provide also a great class of examples, nonexamples and testing objects for many conjectures, cf. [2], recall just a famous Kodaira-Turston nilmanifold being the example of a compact symplectic but nonKähler manifold. From these reasons the problem of classifying which Lie algebras admit an integrable complex structure is not only interesting in it's own right but has also a profound impact on our knowledge on many related objects.

First general results on existence of integrable complex structures on Lie algebras are probably due to Samelson, cf. [10], he proved that any compact Lie algebra, here it means having a compact associated Lie group, admits such a structure. Later it was proved by Morimoto, cf. [11], that the same is true for reductive Lie algebras. Going into the classification by dimensions we have that the problem is trivial for the real dimension two since then any complex structure is integrable. For the dimension four the problem was solved by Snow, cf. [12], and Ovando, cf. [7], for a solvable algebras. With a little work one can extend their results to a full classification in the dimension four since there are very few four dimensional nonsolvable Lie algebras, cf. [8]. In the dimension six the problem is much harder and there are couple reasons for that. Firstly, the computational complexity is much bigger. Secondly, there is no classification of six dimensional Lie algebras, although the classification is know up to dimension five, cf. [8], which coses a lot of problems since most attempts to the question of existence of integrable complex structures rely on case by case study. The classification in dimension six is known for special complex structures, so called abelian complex structures, cf. [1], and for nilpotent algebras, cf. [9]. The work of Salamon was recently refined in [3] by computing moduli spaces of integrable complex structures for six dimensional nilpotent algebras. In [5] we have classified which Lie algebras of the form $\mathfrak{g} \times \mathfrak{g}$, for a three dimensional Lie algebra $\mathfrak{g}$, admit an integrable complex structure. This is a step toward a classification in dimension six at leas for decomposable algebras.

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