Bäcklund theorem for surfaces in the Galilean space G_3

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The 3-dimensional Galilean geometry is the pair $(\mathbf{R}^3, \mathcal{G})$, where $\mathcal{G}$ is the 6-parameter group of transformations which have the form

$$\begin{aligned}
\overline{x} &= x + a, \\
\overline{y} &= bx + \cos\varphi\, y + \sin\varphi\, z + c, \\
\overline{z} &= ex - \sin\varphi\, y + \cos\varphi\, z + f
\end{aligned}$$

if we choose the coordinates x, y, z in a standard way, that is, Euclidean planes correspond to $x = \text{const}$. This geometry can be thought of as the geometry of classical kinematics in the Euclidean (y, z) plane with the time variable x.

The length $|v|$ of a vector $v = (X, Y, Z)$ is equal to $|X|$ if $X \neq 0$ and $\sqrt{Y^2 + Z^2}$ if $X = 0$. The vectors with $X = 0$ are called *isotropic*. The scalar product of isotropic vectors is defined as the restriction of the standard scalar product of $\mathbf{R}^3$ to the Euclidean plane. Having the scalar product we can measure the angle between two isotropic vectors.

In the local theory of surfaces in the Galilean space one considers only surfaces which have no Euclidean tangent planes. Such surfaces are called *admissible*. On an admissible surface one can define – up to sign – the Galilean normal vector field $\mathbf{n}$ (isotropic, perpendicular to the unique isotropic tangent direction $\mathbf{R}\sigma$ and of unit length). By the Gauss formula with $\mathbf{n}$ in the place of a transversal field one defines a linear connection ∇ and the second fundamental form of a surface. The *Gaussian curvature* K and the *mean curvature* H are also defined in the Galilean geometry. A surface is called *minimal* if $H \equiv 0$.

A non-flat Galilean connection ∇ always satisfies the condition $\dim \operatorname{im} R = 1$. One can prove that every surface f with locally symmetric Blaschke connection satisfying the condition $\dim \operatorname{im} R = 1$ is a surface of constant Gaussian curvature in the Galilean space G_3. (If we use the standard coordinates in G_3, then we have to compose f with some affine isomorphism of $\mathbf{R}^3$.)

One can also prove some Galilean versions of the Bäcklund theorem. We assume that $f, \widehat{f} : M \to G_3$ are non-degenerate admissible immersions such that $f_*(T_pM) \cap \widehat{f}_*(T_pM) = \mathbf{R}(\widehat{f}(p) - f(p))$ for every $p \in M$.

Theorem 1. *If $\widehat{f} - f$ is everywhere non-isotropic, of constant (Galilean) length $|\widehat{f} - f| = L$, and the angle α between the Galilean normal vector fields $\mathbf{n}$ and $\widehat{\mathbf{n}}$ is constant with $\sin\alpha \neq 0$, then both surfaces are of constant negative Galilean Gausian curvature $K = \widehat{K} = -\frac{\sin^2\alpha}{L^2}$. The second fundamental forms h and $\widehat{h}$ are proportional.*

Theorem 2. *If $\mathbf{n} \,||\, \widehat{\mathbf{n}}$, then f and $\widehat{f}$ are minimal.*

References

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[2] O. Röschel, *Die Geometrie des Galileischen Raumes*, Bericht der mathematisch-statistischen Sektion in der Forschungsgesselschaft Joanneum, Bericht Nr. 256; (1985? 1986?).