On infinitesimal harmonic transformations of complete Riemannian manifolds

Sergey Stepanov (Math. Dept., Finance University, Moscow, Russia)
E-mail: s.e.stepanov@mail.ru
Irina Tsyganok (Math. Dept., Finance University, Moscow, Russia)
E-mail: i.i.tsyganok@mail.ru

A vector field X on a Riemannian manifold (M, g) is called an *infinitesimal harmonic transformation* of (M, g) if X generates a flow which is a local one-parameter group of harmonic diffeomorphisms (see [1]). We recall that the *kinetic energy* $E(X)$ of the flow generated by X (see [2, p. 2]) is determined by the following equation $E(X) = 2^{-1} \int_M \|X\|^2 \, dVol_g$. Then we have the following

Theorem. *Let* (M, g) *be a complete Riemannian manifold* (M, g) *with nonpositive Ricci curvature. Then all possible infinitesimal harmonic transformations with finite kinetic energy are parallel. If the volume of* (M, g) *is infinite or the Ricci curvature is negative at some point, then every infinitesimal harmonic transformation with finite kinetic energy is identically zero.*

If g is a complete Riemannian metric, X is a vector field, and λ is a constant value on a manifold , then $(\mathrm{g}, \mathrm{X}, \lambda)$ is called a *Ricci soliton* if the Ricci tensor Ric of the metric g satisfies the equation $-2\mathrm{Ric} = \mathrm{L}_X g + 2\lambda g$. In [3], it was shown that a vector field X of the Ricci soliton $(\mathrm{g}, \mathrm{X}, \lambda)$ is an infinitesimal harmonic transformation of a Riemannian manifold(M, g). Then the following corollary holds.

Corollary. *Let* $(\mathrm{g}, \mathrm{X}, \lambda)$ *be a Ricci soliton with complete Riemannian metric g on a manifold . If the Ricci curvature of g is nonpositive and the kinetic energy of the flow generated by Xis finite, then g is a Ricci-flat metric. If, in addition, the volume of M which is defined by g is infinite then X is identically zero.*

These two statements are complementary to our results in [5].

Our work was supported by RBRF grant 16-01-00053-a (Russia).

References

[1] Stepanov S.E., Shandra I.G., Geometry of infinitesimal harmonic transformation, Annals of Global Analysis and Geometry 24 (2003), 291-29.
[2] Arnold V.I., Kresin B.A., Topological methods in hydrodynamics, Springer-Verlag, StateNew York (1998).
[3] Chow B., Knopf D., The Ricci flow: An introduction, American Mathematical Society, Providence (2004).
[4] Stepanov S.E., Shelepova V.N., A note on Ricci soliton, Mathematical Notes 86: 3 (2009), 447-450.
[5] Stepanov S.E., Tsyganok I.I., Infinitesimal harmonic transformations and Ricci solitons on complete Riemannian manifolds, Russian Mathematics 54:3 (2010), 84-87.