Attainability of Hardy inequality through sharp Sobolev-Lorentz embeddings

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It is well known that the classical Hardy inequality

$$\left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \leq \int_\Omega |\nabla u|^p dx$$

is equivalent to the sharp inequality

$$\|u\|_{p^*,p} \leq \frac{p}{n-p}\,\omega_n^{-\frac{1}{n}}\,\|\nabla u\|_p \quad 1 \leq p < n \tag{1}$$

related to the optimal embedding for the Sobolev space $\mathcal{D}^{1,p}(\mathbb{R}^n)$ into Lorentz spaces. It was proved by Alvino together with its generalization in the context of the Sobolev-Lorentz spaces, $\mathcal{D}^1 L^{p,q}(\mathbb{R}^n)$, only in the case $1 \leq q \leq p$; a non-sharp inequality was later obtained by Talenti for any $1 \leq q \leq \infty$.

We will extend here the validity of (1) for any $p \leq q \leq \infty$ *as a consequence of* (1), that is, of Hardy inequality, observing that the sharp constant (which does not depend on q and is strictly related to Hardy's constant) is never attained if $q < \infty$. Surprisingly, in the case $q = \infty$ we will prove the reverse implication, so obtaining the equivalence between Hardy inequality and optimal embedding inequality for Sobolev-Marcinkiewicz space. We will finally prove that the equivalent version of Hardy inequality

$$\|v\|_{p^*,\infty} \leq \frac{p}{n-p}\,\omega_n^{-\frac{1}{n}}\,\|\nabla v\|_{p,\infty}$$

is actually attained, and a maximizer is given by $v(x) = |x|^{-\frac{n-p}{p}}$.

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