Steady states for predator-prey model with diffusion and indirect prey-taxis.

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Emerging issues in nonlinear elliptic equations: singularities, singular perturbations and non local problems
The talk is based on a joint work with Jose Ignacio Tello and Arturo Hidalgo (Politecnica de Madrid)
Inter-species interactions

$u, v$ – densities of two species

\[ u_t = f(u, v), \]
\[ v_t = g(u, v) \]

$u(0) = u_0 > 0, \quad v(0) = v_0 > 0$

Reaction diffusion system in $\Omega \times (0, \infty)$

\[ u_t = \text{div}(d_u \nabla u) + f(u, v), \]
\[ v_t = \text{div}(d_v \nabla v) + g(u, v) \]

$u(0) = u_0 > 0, \quad v(0) = v_0 > 0 + \text{boundary cond. on } \partial \Omega \times (0, \infty)$
We extend the model of interspecies interactions by taking into account the case of chemical signalling.

- Individuals of one population e.g. predator detect an olfactory and diffusive chemical (chemical signal) which is released by prey.

- The gradient of concentration of the chemical indicates the direction of increasing prey density.
We extend the model of interspecies interactions by taking into account the case of chemical signalling.

- Individuals of one population, e.g., predator, detect an olfactory and diffusive chemical (chemical signal) which is released by prey.
- The gradient of concentration of the chemical indicates the direction of increasing prey density.
$w$ -density of some chemical produced by species ν. We consider in $\Omega \times (0, \infty)$

\[
\begin{align*}
    u_t &= \text{div}(d_u \nabla u - \chi u \nabla w) + f(u, v), \\
    w_t &= d_w \Delta w - \lambda w + \alpha v, \\
    v_t &= \text{div}(d_v \nabla v) + g(u, v), \\
\end{align*}
\]

$u(0) = u_0 > 0$, $v(0) = v_0 > 0$ + boundary cond. on $\partial \Omega \times (0, \infty)$
Keller-Segel(’70) model of chemotaxis

Chemical signal is released and detected by individuals of the same population.

Self-attraction;—-blow-up versus global-in-time existence of solutions

\[ u_t = \text{div}\{d_u \nabla -/ + u \nabla w\}, \]
\[ w_t = d_w \Delta w - \lambda w + \alpha u, \]

chemotaxis

chemorepulsion
Predator-prey model,

Logistic equation for prey

\[ v(t) \] – population density at time \( t \geq 0 \)
\( \lambda \) - birth rate
\( K \) – caring capacity

\[ v_t = \lambda v(t) \left( 1 - \frac{v(t)}{K} \right) \]

\( F \) - consumption rate (mortality)

\[ v_t = \lambda v(t) \left( 1 - \frac{v(t)}{K} \right) - Fv(t) \]

If \( \lambda > F \) there is the unique stable steady state

\[ \bar{v} = K \left( 1 - \frac{F}{\lambda} \right) \]
Two Models of indirect prey taxis (IPT)

Predator searching strategy is the superposition of random dispersion and directed movement towards the gradient of some chemical:

**Model IPT1**: released by injured prey during capturing.

**Model IPT2**: released by prey itself ”smell of prey”.

Model IPT1

\[ u \text{– predator density} \]
\[ v \text{– prey density} \]
\[ w \text{– chemical released by injured prey (chemoattractant)} \]

non-dimensionalized version of IPT1 model:

\[
\begin{align*}
    u_t &= \Delta u - \text{div}(\chi u \nabla w), & x \in \Omega, \ t > 0 \\
    w_t &= d_w \Delta w - w + \alpha v F(u), & x \in \Omega, \ t > 0 \\
    v_t &= \lambda v (1 - v) - v F(u), & x \in \Omega, \ t > 0
\end{align*}
\]

with mortality of prey caused by predator

\[ F(u) = \frac{F_m u}{1 + u} \]

no-flux boundary conditions
and nonnegative initial data
Model IPT2

$w$— chemical released by prey "smell of prey"" (chemoattractant)

\[
\begin{align*}
  u_t &= \Delta u - \text{div}(\chi u \nabla w), & x \in \Omega, & t > 0 \\
  w_t &= d_w \Delta w - \mu w + \alpha v, & x \in \Omega, & t > 0 \\
  v_t &= \lambda v (1 - v) - vF(u), & x \in \Omega, & t > 0
\end{align*}
\]
Existence of solutions

\[ W_N^{2,p}(\Omega) = \{ w \in W_N^{2,p}(\Omega) : \frac{\partial w}{\partial \nu}(x) = 0, \quad x \in \partial \Omega \} \]

**Theorem**

Assume that initial functions are nonnegative and for \( p > n \), \( u_0, v_0 \in W^{1,p}(\Omega) \) and \( w_0 \in W_N^{2,p}(\Omega) \). Then there exist a unique solution \((u, w, v)\) to IPT models such that

\[ u, v \in C([0, \infty); W^{1,p}) \quad \text{and} \quad w \in C([0, \infty); W_N^{2,p}). \]

Moreover, for any \( T > 0 \), \( u, w \in C_{x,t}^{2,1}(\Omega \times (0, T)) \) and

\[ u, w, v \geq 0 \quad \text{on} \quad \Omega \times (0, T). \]

Proof is based on Banach fixed point theorem applied for integral formulation of the problem and theory of analytical semigroups.
Steady states and linearization

It follows from the non-flux boundary condition that

\[
\langle u(t) \rangle := \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx = \langle u_0 \rangle := \frac{M}{|\Omega|}, \quad \text{for} \quad t > 0.
\]

If \( F(\bar{u}) < \lambda \) in each of the models there is only one constant steady state with positive components:

for IPT1; \( P^1_1 = (\bar{u}, \bar{w}, \bar{v}) \) with

\[
\bar{u} = \langle u_0 \rangle, \quad \bar{w} = \frac{\alpha}{\mu} \left( 1 - \frac{F(\bar{u})}{\lambda} \right) F(\bar{u}), \quad \bar{v} = 1 - \frac{F(\bar{u})}{\lambda},
\]

for IPT2; \( P^2_1 = (\bar{u}, \bar{w}, \bar{v}) \)

\[
\bar{u} = \langle u_0 \rangle, \quad \bar{w} = \frac{\alpha}{\mu} \left( 1 - \frac{F(\bar{u})}{\lambda} \right), \quad \bar{v} = 1 - \frac{F(\bar{u})}{\lambda}.
\]
There is also a trivial steady state $P_0$ for both models:

$$\bar{u} = \langle u \rangle, \quad \bar{w} = \bar{v} = 0.$$ 

which is a unique space homogeneous steady state provided $F(\langle u_0 \rangle) \geq \lambda$. 

Linearization and the eigenvalue problem

Linearization at a homogeneous steady state \((\bar{u}, \bar{w}, \bar{v})\) to Model IPT1 leads to the following eigenvalue problem

\[
\begin{align*}
\Delta \varphi - \chi \bar{u} \Delta \psi &= \sigma \varphi, \\
 d_w \Delta \psi - \mu \psi + \alpha \bar{v} F'(\bar{u}) \varphi + \alpha F(\bar{u}) \eta &= \sigma \psi, \\
 -\bar{v} F'(\bar{u}) \varphi + (F(\bar{u}) - \lambda) \eta &= \sigma \eta
\end{align*}
\]

where \((\phi, \psi, \eta) \in X_0 \times X \times Y\) and

\[
X_0 = \{ \varphi \in W^{2,p}(\Omega) : \frac{\partial \phi}{\partial \nu} = 0, \int_{\Omega} \phi(x) dx = 0 \}, \\
X = \{ \varphi \in W^{2,p}(\Omega) : \frac{\partial \psi}{\partial \nu} = 0 \}, \quad Y = L^2(\Omega).
\]
Let \( \{\lambda_n\}_{n=0}^{\infty} \) be the sequence of eigenvalues of operator \(-\Delta\) with homogeneous Neumann boundary conditions defined on \(X\)

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots
\]

and \( \{w_n\}_{n=0}^{\infty} \) the corresponding system of orthonormal eigenfunctions in \(L^2(\Omega)\). Let us define matrix

\[
A_n = \begin{bmatrix}
-\lambda_n & \chi \bar{u} \lambda_n & 0 \\
\alpha \bar{v} f_1 & -(d_w \lambda_n + \mu) & -\alpha f \\
\bar{v} f_2 & 0 & r
\end{bmatrix}
\]

where \( f = F(\bar{u}) \), \( r = \lambda - f - 2\bar{v} \lambda \) and \( f_1 = f_2 = F'(\bar{u}) \) in Model IPT1 and \( f_1 = 0 \), \( f_2 = F'(\bar{u}) \) in the case of Model IPT2.

**Proposition**

A complex number \( \sigma \) is an eigenvalue to the linearized system iff there exists \( n \geq 1 \) such that \( \sigma \) is an eigenvalue of matrix \( A_n \) or for \( n = 0 \), \( \sigma \in \{-\mu, r\} \). Moreover spectrum of the linear operator consist only of eigenvalues.
Stability criterion

Theorem

Steady state $P_1^1$ in Model IPT1 is locally asymptotically stable if

$$\frac{\chi \alpha \bar{u} F'(\bar{u})}{\lambda} < (1 + d_w) \min \left( \frac{2\mu}{\mu + F(\bar{u})}, 1 \right).$$

Steady state $P_1^2$ in Model IPT2 is locally asymptotically stable if

$$\frac{\chi \alpha \bar{u} F'(\bar{u})}{\lambda} < (1 + d_w) \frac{2\mu}{F(\bar{u})}.$$

In each of the above cases there exists $\delta_0 > 0$ such that if $\sigma$ is an eigenvalue to $A_n$ then $\Re \sigma < -\delta_0 < 0$.

Steady state $P_0$ is unstable provided $\lambda \geq F(\bar{u})$. There is $K > 0$ such that steady states $P_1^1$ and $P_1^2$ are unstable provided

$$\frac{\chi \alpha \bar{u}}{\lambda} > K.$$
The stationary problem for IPT1 may be reduced to the system of two elliptic equations:

\[ \Delta u - \text{div}(u \chi \nabla w) = 0, \quad x \in \Omega, \]
\[ d_w \Delta w - \mu w + \alpha \left(1 - \frac{F(u)}{\lambda}\right) F(u) = 0, \quad x \in \Omega. \]

with homogeneous Neumann boundary conditions.
Let us denote

\[ \Gamma(u) = \left(1 - \frac{F(u)}{\lambda}\right) F(u) \]

and

\[ \gamma := \Gamma'(\bar{u}) = \left(1 - \frac{2F(\bar{u})}{\lambda}\right) F'(\bar{u}). \]

Linearization at the constant steady state \((\bar{u}, \bar{w})\) leads to the following eigenvalue problem \((L)\)

\[
\begin{align*}
\Delta \phi - \chi \bar{u} \Delta \psi & = \sigma \phi, \\
dl_w \Delta \psi - \mu \psi + \alpha \gamma \phi & = \sigma \psi.
\end{align*}
\]
Linearization

Let us define matrix

\[ B_n = \begin{bmatrix} -\lambda_n, & \chi \bar{u} \lambda_n \\ \alpha \gamma, & -\lambda_n d_w - \mu \end{bmatrix} \]

Proposition

A complex number \( \sigma \) is an eigenvalue iff there exists \( n \geq 1 \) such that \( \sigma \) is the eigenvalue to matrix \( B_n \) or \( \sigma = -\mu \). Moreover, \( \text{Re} \ \sigma < 0 \) iff

\[ \lambda_1 > \frac{\alpha \gamma \chi \bar{u} - \mu}{d_w}. \]
If $\gamma > 0$ then it is convenient to choose $\chi$ as a bifurcation parameter and then we obtain the stability condition for the reduced problem

$$\chi < \chi_1 := \frac{\lambda_1 d_w + \mu}{\gamma \alpha \bar{u}}.$$ 

**Proposition**

Assume 1-D case and $\gamma > 0$, $M > 0$. Then for any $\chi > \chi_1$ there are non-constant steady states with the mass $M$.

we adapt result by Xuefeng Wang and Quian XU (2013)
Bifurcation theory is applied to mapping
\[ \Psi : X^2 \times \mathbb{R} \rightarrow Y_0 \times Y \times \mathbb{R} \]

\[ \Psi(u, w; \chi) = \begin{pmatrix}
  \text{div}(\nabla u - u \chi \nabla v) \\
  d_w \Delta w - \mu w + \alpha \left(1 - \frac{F(u)}{\lambda}\right) F(u) \\
  \int_{\Omega} u(x)dx - |\Omega| \bar{u}
\end{pmatrix} \]

and we consider equation

\[ \Psi(u, w; \chi) = 0 \]

knowing that

\[ \Psi(\bar{u}, \bar{w}; \chi) = 0 \]
Comparison of models IPT1 and IPT2

The constant solution to the reduced problem (IPT2) is uniquely determined. Indeed

$$\text{div}\{u \nabla (\ln u - \chi w)\} = 0$$

thus there is $\varrho > 0$ such that

$$u = \varrho e^{\chi w}.$$

Then any non-zero steady state satisfies the following semilinear elliptic equation

$$-\Delta w + \mu w + R(w) = 0 \quad \text{on} \quad \Omega$$

with no-flux boundary condition and

$$R(w) = \alpha \left( \frac{F(\varrho e^{\chi w})}{\lambda} - 1 \right)$$

Uniqueness of solutions results from the fact that

$$w \rightarrow \mu w + R(w)$$

is a strictly increasing function and classical arguments may be applied.

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Assumptions:

\[ \lambda > F_m, \]

there exists a positive constant \( v_0 > 0 \) such that the initial data \( v_0 \) satisfies

\[ v_0 \leq v_0(x) \leq 1 \]

\[ w_0(x) \geq 0, \]

\[ M < \frac{2^7}{3^3} \left( \frac{\chi^2 \alpha^2 F_m^2 |\Omega|^2}{2d_w} \left( 1 + \frac{F_m^2}{\lambda \min\{\lambda v_0, (\lambda - F_m)\}} \right) \right)^{-1}. \]
Theorem
Under assumptions above if $\Omega \subset \mathbb{R}^2$ is a bounded domain

\[ u(\cdot, t) \rightarrow \bar{u} = \frac{1}{|\Omega|} \int_\Omega u \quad \text{in } L^2(\Omega) \text{ as } t \to \infty, \]
\[ v(\cdot, t) \rightarrow \bar{v} = 1 - \frac{1}{\lambda} F(\bar{u}) \quad \text{in } L^p(\Omega) \text{ as } t \to \infty, \]
\[ w(\cdot, t) \rightarrow \bar{w} = \alpha(1 - \frac{1}{\lambda} F(\bar{u}) F(\bar{u})) \quad \text{in } L^p(\Omega) \text{ as } t \to \infty, \]

for any $p \in [1, \infty)$

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crucial energy estimate

\[ \int_{\Omega} u (\ln u - 1) + \int_{\Omega} |\nabla w|^2 + \int_{\Omega} \left| \frac{\nabla v}{v} \right|^2 + \int_0^\infty \int_{\Omega} |\nabla w|^2 + \int_0^\infty \int_{\Omega} \frac{|\nabla u|^2}{1+u} \leq C. \]
first step

\[ \| u(\cdot, t) - \frac{1}{|\Omega|} \int_{\Omega} u(x, t) \, dx \|_p \to 0 \]

\[ \| v(\cdot, t) - \frac{1}{|\Omega|} \int_{\Omega} v(x, t) \, dx \|_p \to 0 \]

\[ \| w(\cdot, t) - \frac{1}{|\Omega|} \int_{\Omega} w(x, t) \, dx \|_p \to 0 \]

as \( t \to +\infty \).
Numerical solutions 1

\[ L = 1, \chi = 1, n = 8, d_w = 0.5, d_u = 1, \lambda = 4, k = 1, \alpha = 100 \]

\[ u(x, 0) = 5 + \cos(n\pi x), \quad w(x, 0) = 0, \quad v(x, 0) = 21 + 20 \cos(n\pi x) \]
Numerical solutions 2

\[ L = 1, \chi = 1, n = 1, d_w = 0.5, d_u = 1, \lambda = 4, k = 1, \alpha = 100 \]

\[ u(x, 0) = 5 + \cos(n\pi x), \quad w(x, 0) = 0, \quad v(x, 0) = 21 + 20 \cos(n\pi x) \]
\[ L = 1, \chi = 5, n = 2, d_W = 0.5, d_U = 1, \lambda = 4, k = 1, \alpha = 100 \]
\[ u(x, 0) = 5 + \cos(n\pi x), \quad w(x, 0) = 0, \quad v(x, 0) = 21 + 20 \cos(n\pi x) \]
$L = 1, \chi = 5, d_w = 0.5, d_u = 1, \lambda = 4, k = 1, \alpha = 100$
Where do the patterns come from?

From the first equation \( u = \varrho e^{\chi w} \) and

\[
d_w w'' = \mu w - \alpha F(\varrho e^{\chi w}) + \alpha F^2(\varrho e^{\chi w})
\]

Suppose for simplicity \( F(u) = F_m u \). Then

\[
d_w w'' = -\mathcal{U}(w)
\]

with potential

\[
\mathcal{U}(w) = -\frac{\mu}{2} w^2 + \frac{\alpha F_m \varrho}{\chi} e^{\chi w} - \frac{\alpha F_m^2 \varrho^2}{2\lambda \chi} e^{2\chi w}
\]

For some range of parameters \( \mathcal{U} \) has a well.
Where do the patterns come from?

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