The Fredholm Alternative for the p-Laplacian in exterior domains. Part I

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KNOWN RESULTS

- Let Ω be a bounded domain in $\mathbb{R}^N$ and let (λ_1, φ_1) be the first eigenpair of $(-\Delta, W_0^{1,2}(\Omega))$ and let $h \in L^2(\Omega)$. Consider

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(1) \qquad \begin{cases} -\Delta u = \lambda_1 u + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}
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$$\begin{cases} -\Delta u = \lambda_1 u + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Problem (1) is solvable if and only if $\int_\Omega h\varphi_1 = 0$.

INTRODUCTION

- del Pino-Drábek-Manásevich (1999), Drábek-Girg-Manásevich (2001), Drábek-Holubová (2001), Drábek (2002), Takáč (2002, 2006) etc. :

$$(2)\qquad \begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2} u + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

  where $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian with $p > 1$, Ω is a bounded domain of $\mathbb{R}^N$.

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Let $1 < p \neq 2$. For a given $h^\perp \in L^\infty(\Omega)$ with $\int_\Omega h^\perp \varphi_1 = 0$, there is $\epsilon > 0$ such that problem (2) is solvable if $h = h^\perp + \zeta\varphi_1,\ \zeta \in (-\epsilon, \epsilon)$.

- Alziary-Fleckinger-Takáč (2004):

(3) $$-\Delta_p u = \lambda_1 m(x)|u|^{p-2}u + h \quad \text{in } \mathbb{R}^N, \; u \in \mathcal{D}^{1,p}(\mathbb{R}^N),$$

where $1 < p < N$, the weight $m(x) = m(|x|)$ satisfies

(4) $$0 < m(r) \le \frac{C}{(1+r)^{p+\mu}} \quad \text{a.e. in } [0, \infty).$$

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For a given $h^* \in [\mathcal{D}^{1,p}(\mathbb{R}^N)]^*$ satisfying $\langle h^*, \varphi_1 \rangle = 0$, problem (3) is solvable

(i) *for $2 \leq p < N$ with $h = h^*$;*

(ii) *for $1 < p < 2 \leq N$ with h in a neighbourhood of h^*.*

INTRODUCTION

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$$\text{(P)} \qquad \begin{cases} -\Delta_p u = \lambda_1 K(x)|u|^{p-2}u + h & \text{in } B_1^c, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where B_1^c is the complement of the closed unit ball B_1 in $\mathbb{R}^N$ $(N \geq 2)$, λ_1 is the first eigenvalue of $-\Delta_p$ in B_1^c relative to the weight K.

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We study (P) with an *admissible* weight K:

(i) $K \in L^1_{loc}(B_1^c)$, $\text{meas}\{x \in B_1^c : K(x) > 0\} > 0$;

(ii) $|K(x)| \leq w(|x|)$ for a.e. $x \in B_1^c$;

(iii) $0 < w \in \begin{cases} L^1((1,\infty); r^{p-1}), & p \neq N, \\ L^1((1,\infty); [r \log r]^{N-1}), & p = N. \end{cases}$

THE EIGENVALUE PROBLEM

Anoop-Drábek-Sasi (2015) and Anoop-Drábek-Sankar-Sasi (2016):

$$\text{(E)} \qquad \begin{cases} -\Delta_p u = \lambda K(x)|u|^{p-2}u & \text{in } B_1^c, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

$X :=$ the completion of $C_c^\infty(B_1^c)$ w.r.t. $\|u\| = \left(\int_{B_1^c} |\nabla u|^p dx\right)^{1/p}$. Then

$$\text{(5)} \qquad \lambda_1 := \inf\left\{\int_{B_1^c} |\nabla u|^p dx : \ u \in X, \int_{B_1^c} K(x)|u|^p dx = 1\right\}$$

is a simple eigenvalue of problem (E) with corresponding eigenfunction $\varphi_1 > 0$ a.e. in B_1^c. If in addition,

- $K \in L^\infty(A_1^R)$ for all $R > 1$, where $A_1^R := \{x \in \mathbb{R}^N : 1 < |x| < R\}$, then λ_1 is isolated and $\varphi_1 \in C^{1,\alpha}(\overline{A_1^R})$ for all $R > 1$.

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From now on, we denote by X^* the dual space of X and define

$$X^\perp := \left\{u \in X : \int_{B_1^c} K(x)\varphi_1^{p-1} u dx = 0\right\}.$$

THE GEOMETRY OF THE ENERGY FUNCTIONAL

We will obtain the existence of solutions by using variational arguments but treated the two cases $1 < p < 2 \leq N$ and $2 \leq p < N$ in a different way due to the geometry of the energy functional

$$J_h(u) = \frac{1}{p}\int_{B_1^c} |\nabla u|^p dx - \frac{\lambda_1}{p}\int_{B_1^c} K(x)|u|^p dx - \langle h, u \rangle.$$

- $1 < p < 2 \leq N$: J_h has "a saddle point geometry";
- $2 \leq p < N$: J_h has a "global minimizer geometry" due to an improved Poincaré inequality.

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We apply the second order Taylor formula for J_h at φ_1 :

$$\to \int |\nabla \varphi_1|^{p-2}|\nabla \phi|^2 dx \to \mathcal{A} := \{\nabla \varphi_1(x) = 0\}.$$

- The entire space $\mathbb{R}^N$ case (with radial symmetric weight): $\mathcal{A} = \{0\}$.
- Our case (with radial symmetric weight): $\mathcal{A} = \{|x| = r_0\} \quad (r_0 > 1)$.

A SADDLE POINT GEOMETRY WHEN $1 < p < 2$

Let $1 < p < 2$ and let $h \in X^* \setminus \{0\}$ with $\langle h, \varphi_1 \rangle = 0$. To deal with

$$\int_{B_1^c} |\nabla \varphi_1|^{p-2} |\nabla \phi|^2 dx$$

we look for ϕ such that $\langle h, \phi \rangle \neq 0$ and ϕ belongs to

$$Y := \{\phi \in C_c^1(B_1^c) : \phi \text{ is constant on some neighbourhood of } \mathcal{A}\}.$$

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Lemma 1.

Let $1 < p < N$ and assume that $K(x) = K(|x|) > 0$ for a.e. $x \in B_1^c$, $K \in L^\infty(B_1^c)$ and $K \in L^1((1, \infty); r^\delta)$ for some $\delta \in (p-1, N-1)$. Then there is an $h \in X^* \setminus \{0\}$ such that $\langle h, \varphi_1 \rangle = 0$ and $h \equiv 0$ on Y.

A SADDLE POINT GEOMETRY WHEN $1 < p < 2$

We introduce the condition for the source term h:

$$h \in X^*_Y, \text{ where } X^*_Y := \{h \in X^* : h \not\equiv 0 \text{ on } Y\}.$$

Lemma 2.

Assume that $K > 0$ a.e. in B_1^c and $K \in L^\infty(A_1^R)$ for all $R > 1$. Then X^*_Y is open and dense in X^* and contains the set

$$Z := \left\{h \in X^* \setminus \{0\} : \exists g \in C_c(B_1^c) \text{ s.t. } \langle h, u\rangle = \int_{B_1^c} gudx, \forall u \in X\right\}.$$

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We introduce the condition for the source term h:

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Theorem 1 (a saddle point geometry).

Assume that $1 < p < 2$ and that $K \in L^\infty(A_1^R)$ for all $R > 1$. Let $h \in X_Y^*$ with $\langle h, \varphi_1\rangle = 0$. Then for any $M > 0$, there exist $\tau_0 > 0$, such that for each $\tau > \tau_0$ we can find $v_\pm^\tau \in X^\perp$ such that

$$\max\{J_h(\tau\varphi_1 + v_+^\tau), J_h(-\tau\varphi_1 + v_-^\tau)\} < -M \ (< 0 \le \inf_{v \in X^\perp} J_h(v)).$$

AN IMPROVED POINCARÉ INEQUALITY WHEN $2 < p < N$

Assume that

(H) $K(x) = K(|x|) > 0$ for a.e. $x \in B_1^c$, $K \in L^\infty(B_1^c)$ and $K \in L^1((1,\infty); r^\delta)$ for some $\delta \in (p-1, N-1)$.

(W) $K^{-1} \in L^1_{loc}(1,\infty)$ and for each $t > 1$,

$$f(r) := \left| \int_t^r K(s)ds \right|^{\frac{2-p}{p-1}} \in L^1_{loc}(1,\infty).$$

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Theorem 2 (an improved Poincaré inequality for nonlinear case).

Let $2 < p < N$, (H), (W) and $\lim_{\rho\to\infty} \operatorname{ess\,sup}_{r\geq\rho} r^p K(r) = 0$ hold. Then $\exists C = C(p,K) > 0$ s.t. for all $u = \tau\varphi_1 + u^\perp$ with $\tau \in \mathbb{R}$ and $u^\perp \in X^\perp$,

$$\int_{B_1^c} |\nabla u|^p dx - \lambda_1 \int_{B_1^c} K(x)|u|^p dx$$

$$\geq C\left(|\tau|^{p-2}\int_{B_1^c} |\nabla\varphi_1|^{p-2}|\nabla u^\perp|^2 dx + \int_{B_1^c} |\nabla u^\perp|^p dx\right). \tag{6}$$

A SKETCH PF PROOF OF THEOREM 2

Define $\mathcal{D}_{\varphi_1}$:= the completion of X with respect to the norm

$$\|u\|_{\mathcal{D}_{\varphi_1}} := \left(\int_{B_1^c} |\nabla \varphi_1|^{p-2} |\nabla u|^2 dx \right)^{1/2}.$$

For $\Phi(u) := \frac{1}{p} \int_{B_1^c} |\nabla u|^p dx - \frac{\lambda_1}{p} \int_{B_1^c} K(x)|u|^p dx$, we have

$$\Phi(\varphi_1+\phi) = \Phi(\varphi_1)+\langle D\Phi(\varphi_1), \phi\rangle + \int_0^1 (1-s)\langle D^2\Phi(\varphi_1+s\phi)\phi, \phi\rangle ds = \mathcal{Q}_\phi(\phi, \phi),$$

where

$$\mathcal{Q}_\phi(v, w) := \int_{B_1^c} \left\langle \left[\int_0^1 \mathbb{A}(\nabla\varphi_1 + s\nabla\phi)(1-s)ds \right] \nabla v, \nabla w \right\rangle_{\mathbb{R}^N} dx$$

$$(7) \qquad - \lambda_1(p-1) \int_{B_1^c} K(x) \left[\int_0^1 |\varphi_1 + s\phi|^{p-2}(1-s)ds \right] vwdx,$$

where $\mathbb{A}(\mathbf{a}) := |\mathbf{a}|^{p-2} \left(\mathbf{I} + (p-2)\frac{\mathbf{a}\otimes\mathbf{a}}{|\mathbf{a}|^2} \right)$ for $\mathbf{a} \in \mathbb{R}^N \setminus \{0\}$,
$\mathbf{a} \otimes \mathbf{b} := (a_i b_j)_{N\times N}$ with $\mathbf{a} = (a_1, \cdots, a_N), \mathbf{b} = (b_1, \cdots, b_N) \in \mathbb{R}^N$.

THANK YOU FOR YOUR ATTENTION!