Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a singularity at $u = 0$

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The talk is based on the joint work with Daniela Giachetti (Rome, Italy) and Pedro J. Martínez-Aparicio (Cartagena, Spain)

We consider the following semilinear elliptic equation with a singularity at $u = 0$

$$u \geq 0 \quad \text{in } \Omega,$$

$$-div\, A(x)Du = F(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega,$$

where $F(x,s)$ is a Carathéodory function which satisfies

$$0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)} \quad \text{a.e. } x \in \Omega, \quad \forall s > 0,$$

with h in some $L^r(\Omega)$ and Γ a $C^1([0,+\infty[)$ function such that $\Gamma(0) = 0$ and $\Gamma'(s) > 0$ for every $s > 0$.

In the case where the singularity is mild, i.e. when

$$0 \leq F(x,s) \leq h(x)(\frac{1}{s} + 1) \quad \text{a.e. } x \in \Omega, \quad \forall s > 0,$$

we define the solution as a function $u \in H_0^1(\Omega)$ which satisfies the equation for test functions $v \in H_0^1(\Omega)$. We prove the existence of such a solution and its stability with respect to variations of $F(x,s)$, as well as its uniqueness when $F(x,s)$ is nonincreasing in s. A key ingredient in the proof is to prove that the integral $\displaystyle\int_{\{u \leq \delta\}} F(x,u)\, v$ tends to zero in a controlled way as δ tends to zero.

In the case where the singularity is stronger, the solution in general does not belongs to $H_0^1(\Omega)$ anymore. We then introduce a notion of solution which is more sophisticated: the solution is required to belong to the class of those functions such that $u \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega)$, $u \geq 0$, $G_k(u) \in H_0^1(\Omega)$ and $\varphi T_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, where as usual $G_k(s) = (s-k)^+$ and $T_k(s) = \inf\{s,k\}$ for $s > 0$, while the equation has to be satisfied for a non standard class of nonnegative test functions, in the spirit of the notion of solutions defined by transposition. This definition allows us to perform, mutatis mutandis, the various computations that we made in the case of mild singularities, and to prove the existence of such a solution and its stability with respect to variations of $F(x,s)$, as well as its uniqueness when $F(x,s)$ is nonincreasing in s.