Tikhonov regularization with oversmoothing penalty

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Abstract

We consider a (nonlinear) operator equation $F(x^\dagger) = y$ where $F\colon \mathcal{D}(F) \subset X \to Y$ is a mapping acting between Hilbert spaces. In presence of noisy data $\|y - y^\delta\| \leq \delta$ regularization is required in order to find a suitable reconstruction $x = x(\delta, y^\delta)$. A versatile means is *Tikhonov regularization*, where a penalty $\mathcal{R}$ is added, i.e., the approximate solution, say x_α^δ, is obtained as a minimizer of the Tikhonov functional $T_\alpha(x) := \|F(x) - y^\delta\|^2 + \alpha\mathcal{R}(x)$. In addition we shall assume that the parameter $\alpha = \alpha(\delta, y^\delta)$ is chosen according to the *discrepancy principle* from $\|F(x_\alpha^\delta) - y^\delta\| = C\delta$.

The choice of the penalty has impact on the features of the reconstruction. In different applications, various penalties have proved successful (TV-, l_1-, L_1-penalties). We shall analyze *smoothing penalties*, i.e, when these are given through some (unbounded linear self-adjoint) operator $B\colon \mathcal{D}(B) \subset X \to X$ as $\mathcal{R}(x) := \|B(x - \bar{x})\|^2,\ x \in \mathcal{D}(B)$, and $\bar{x}$ is a chosen reference element. Thus we analyze properties of the minimizers of

$$T_\alpha(x) := \|F(x) - y^\delta\|^2 + \alpha\|B(x - \bar{x})\|^2.$$

In particular we ask whether the minimizers x_α^δ tend to the solution $x^\dagger$ as $\delta \to 0$.

The answer was known in two cases. First, if the true solution belongs to the domain of B, see e.g. [2], and secondly, if the mapping F is linear, see [1]. We discuss that the answer to the above question is affirmative for certain nonlinear mappings F and in the oversmoothing case ($x^\dagger \notin \mathcal{D}(B)$).

References

[1] F. Natterer. Error bounds for Tikhonov regularization in Hilbert scales. *Applicable Anal.*, 18(1-2):29–37, 1984.

[2] T. Schuster, B. Kaltenbacher, B. Hofmann, and K. S. Kazimierski. *Regularization Methods in Banach Spaces*, volume 10 of *Radon Series on Computational and Applied Mathematics*. Walter de Gruyter, Berlin/Boston, 2012.

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