# Simple Bilevel Programming and Extensions: Theory and Algorithms

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## Abstract

Let $S =$ argmin $\{h(x) : x \in C\}$ denote the solution set of a convex optimization problem, where $C \subset \mathbb{R}^n$ is a closed convex set and $f, h : \mathbb{R}^n \to \mathbb{R}$ are real-valued convex functions. The simple bilevel optimization problem is

$$\min\{f(x) : x \in S\}. \tag{1}$$

To investigate this convex optimization problem, the lower level problem needs to be transformed. If $\alpha = \min\{h(x) : x \in C\}$ denote the optimal value of this problem, it is equivalent to $\min\{f(x) : h(x) \leq \alpha,\ x \in C\}$. Slater's regularity condition is violated for this convex optimization problem. Using a variational inequality to express the set $S$, a simple MPEC

$$\min\{f(x) : x \in C,\ \psi_y(x) \leq 0\ \forall\ y \in C\} \tag{2}$$

arises, where $\psi_y(x) = \langle \nabla h(y), x - y\rangle$. (2) is a convex optimization problem. Results from semi-infinite optimization or the use of a gap function for the variational inequality can be applied to derive necessary and sufficient optimality conditions for (2) and, hence, for the original problem. One such optimality condition reads as:

A feasible point $\bar{x}$ is optimal for (2) if and only if there exist $k \in \mathbb{N}$, $\lambda_1, \lambda_2, \cdots, \lambda_k > 0$, $y_1, y_2, \cdots, y_k \in C$ such that

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{k} \lambda_i \nabla h(y_i) + N_C(\bar{x}) \text{ and} \langle \nabla h(y_i), \bar{x} - y_i\rangle = 0, \text{ for all } i = 1, 2, \cdots, k$$

provided some closedness qualification condition is satisfied and $\nabla h(x)$ is continuous and monotone.

In the second part of the talk, an idea for solving the problem will be given. Basis for this algorithm is a penalization $\xi_\varepsilon(x) = h(x) + \varepsilon f(x)$, where both $f, h$ are assumed to be convex but not necessarily differentiable. The algorithm computes a sequence of $\eta_k$-optimal solutions of minimizing a Moreau-Yosida regularization of the function $\xi_\varepsilon(x)$ over $C$ which converges to to an optimal solution provided this problem has a solution.

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