# WORST SINGULARITIES OF PLANE CURVES OF GIVEN DEGREE

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ABSTRACT. We prove that $\frac{2}{d}$, $\frac{2d-3}{(d-1)^2}$, $\frac{2d-1}{d(d-1)}$, $\frac{2d-5}{d^2-3d+1}$ and $\frac{2d-3}{d(d-2)}$ are the smallest log canonical thresholds of reduced plane curves of degree $d \geqslant 3$, and we describe reduced plane curves of degree $d$ whose log canonical thresholds are these numbers. As an application, we prove that $\frac{2}{d}$, $\frac{2d-3}{(d-1)^2}$, $\frac{2d-1}{d(d-1)}$, $\frac{2d-5}{d^2-3d+1}$ and $\frac{2d-3}{d(d-2)}$ are the smallest values of the $\alpha$-invariant of Tian of smooth surfaces in $\mathbb{P}^3$ of degree $d \geqslant 3$. We also prove that every reduced plane curve of degree $d \geqslant 4$ whose log canonical threshold is smaller than $\frac{5}{2d}$ is GIT-unstable for the action of the group $\mathrm{PGL}_3(\mathbb{C})$, and we describe GIT-semistable reduced plane curves with log canonical thresholds $\frac{5}{2d}$.



All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

## 1. Introduction

Let $C_d$ be a *reduced* plane curve in $\mathbb{P}^2$ of degree $d \geqslant 3$, and let $P$ be a point in $C_d$. The curve $C_d$ can have *any* given plane curve singularity at $P$ provided that its degree $d$ is *sufficiently* big. Thus, it is natural to ask

**Question 1.1.** What is the *worst* singularity that $C_d$ can have at $P$?

Denote by $m_P$ the multiplicity of the curve $C_d$ at the point $P$, and denote by $\mu(P)$ the Milnor number of the point $P$. If we use $m_P$ to measure the singularity of $C_d$ at the point $P$, then a union of $d$ lines passing through $P$ is an answer to Question 1.1, since $m_P \leqslant d$, and $m_P = d$ if and only if $C_d$ is a union of $d$ lines passing through $P$. If we use the Milnor number $\mu(P)$, then the answer would be the same, since $\mu(P) \leqslant (d-1)^2$, and $\mu(P) = (d-1)^2$ if and only if $C_d$ is a union of $d$ lines passing through $P$. Alternatively, we can use the number

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \sup\Big\{\lambda \in \mathbb{Q} \ \Big| \ \text{the log pair } \big(\mathbb{P}^2, \lambda C_d\big) \text{ is log canonical at } P\Big\},$$

which is known as the *log canonical threshold* of the log pair $(\mathbb{P}^2, C_d)$ at the point $P$ or the log canonical threshold of the curve $C_d$ at the point $P$ (see [4, Definition 6.34]). The smallest $\mathrm{lct}_P(\mathbb{P}^2, C_d)$ when $P$ runs through all points in $C_d$ is usually denoted by $\mathrm{lct}(\mathbb{P}^2, C_d)$. Note that

$$\frac{1}{m_P} \leqslant \mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) \leqslant \frac{2}{m_P}.$$

This is well-known (see, [4, Exercise 6.18] and [4, Lemma 6.35]). So, the smaller $\mathrm{lct}_P(\mathbb{P}^2, C_d)$, the worse singularity of the curve $C_d$ at the point $P$ is.

**Example 1.2.** Suppose that $C_d$ is given by $x_1^{n_1}x_2^{n_2}(x_1^{m_1} + x_2^{m_2}) = 0$ up to analytic change of local coordinates, where $m_1$ and $m_2$ are non-negative integers, and $n_1, n_2 \in \{0, 1\}$. Then

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \min\left\{1, \frac{\frac{1}{m_1} + \frac{1}{m_2}}{1 + \frac{n_1}{m_1} + \frac{n_2}{m_2}}\right\}$$

by [8, Proposition 2.2].

Log canonical thresholds of plane curves have been intensively studied (see, for example, [8]). Surprisingly, they give the same answer to Question 1.1 by

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**Theorem 1.3** ([1, Theorem 4.1])**.** One has $\mathrm{lct}_P(\mathbb{P}^2, C_d) \geqslant \frac{2}{d}$. Moreover, $\mathrm{lct}(\mathbb{P}^2, C_d) = \frac{2}{d}$ if and only if $C_d$ is a union of $d$ lines that pass through $P$.

In this paper we want to address

**Question 1.4.** What is the *second worst* singularity that $C_d$ can have at $P$?

To give a *reasonable* answer to this question, we have to disregard $m_P$ by obvious reasons. Thus, we will use the numbers $\mu(P)$ and $\mathrm{lct}_P(\mathbb{P}^2, \mathbb{C}_d)$. For cubic curves, they give the same answer.

**Example 1.5.** Suppose that $d = 3$, $m_P < 3$ and $P$ is a singular point of $C_3$. Then $P$ is a singular point of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$. Moreover, if $C_3$ has singularity of type $\mathbb{A}_3$ at $P$, then $C_3 = L + C_2$, where $C_2$ is a smooth conic, and $L$ is a line tangent to $C_2$ at $P$. Furthermore, we have

$$\mu(P) = \begin{cases} 1 \text{ if } C_3 \text{ has } \mathbb{A}_1 \text{ singularity at } P, \\ 2 \text{ if } C_3 \text{ has } \mathbb{A}_2 \text{ singularity at } P, \\ 3 \text{ if } C_3 \text{ has } \mathbb{A}_3 \text{ singularity at } P. \end{cases}$$

Similarly, we have

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_3\big) = \begin{cases} 1 \text{ if } C_3 \text{ has } \mathbb{A}_1 \text{ singularity at } P, \\ \dfrac{5}{6} \text{ if } C_3 \text{ has } \mathbb{A}_2 \text{ singularity at } P, \\ \dfrac{3}{4} \text{ if } C_3 \text{ has } \mathbb{A}_3 \text{ singularity at } P. \end{cases}$$

For quartic curves, the numbers $\mu(P)$ and $\mathrm{lct}_P(\mathbb{P}^2, \mathbb{C}_d)$ give different answers to Question 1.4.

**Example 1.6.** Suppose that $d = 4$, $m_P < 4$ and $P$ is a singular point of $C_4$. Going through the list of all possible singularities that $C_P$ can have at $P$ (see, for example, [6]), we obtain

$$\mu(P) = \begin{cases} 6 \text{ if } C_4 \text{ has } \mathbb{D}_6 \text{ singularity at } P, \\ 6 \text{ if } C_4 \text{ has } \mathbb{A}_6 \text{ singularity at } P, \\ 6 \text{ if } C_4 \text{ has } \mathbb{E}_6 \text{ singularity at } P, \\ 7 \text{ if } C_4 \text{ has } \mathbb{A}_7 \text{ singularity at } P, \\ 7 \text{ if } C_4 \text{ has } \mathbb{E}_7 \text{ singularity at } P, \end{cases}$$

and $\mu(P) < 6$ in all remaining cases. Similarly, we get

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_4\big) = \begin{cases} \dfrac{5}{8} \text{ if } C_4 \text{ has } \mathbb{A}_7 \text{ singularity at } P, \\ \dfrac{5}{8} \text{ if } C_4 \text{ has } \mathbb{D}_5 \text{ singularity at } P, \\ \dfrac{3}{5} \text{ if } C_4 \text{ has } \mathbb{D}_6 \text{ singularity at } P, \\ \dfrac{7}{12} \text{ if } C_4 \text{ has } \mathbb{E}_6 \text{ singularity at } P, \\ \dfrac{5}{9} \text{ if } C_4 \text{ has } \mathbb{E}_7 \text{ singularity at } P, \end{cases}$$

and $\mathrm{lct}_P(\mathbb{P}^2, C_4) > \frac{5}{8}$ in all remaining cases.

Recently, Arkadiusz Płoski proved that $\mu(P) \leqslant (d-1)^2 - \lfloor\frac{d}{2}\rfloor$ provided that $m_P < d$. Moreover, he described $C_d$ in the case when $\mu(P) = (d-1)^2 - \lfloor\frac{d}{2}\rfloor$. To present his description, we need

**Definition 1.7.** The curve $C_d$ is an *even Płoski* curve if $d$ is even, the curve $C_d$ has $\frac{d}{2} \geqslant 2$ irreducible components that are smooth conics passing through $P$, and all irreducible components of $C_d$ intersect each other pairwise at $P$ with multiplicity 4. The curve $C_d$ is an *odd Płoski* curve if $d$ is odd, the curve $C_d$ has $\frac{d+1}{2} \geqslant 2$ irreducible components that all pass through $P$, $\frac{d-1}{2}$ irreducible component of the curve $C_d$ are smooth conics that intersect each other pairwise at $P$ with multiplicity 4, and the remaining irreducible component is a line in $\mathbb{P}^2$ that is tangent at $P$ to all other irreducible components. We say that $C_d$ is *Płoski* curve if it is either an even Płoski curve or an odd Płoski curve.

Each Płoski curve has unique singular point. If $d = 4$, then $C_4$ is a Płoski curve if and only if it has a singular point of type $\mathbb{A}_7$. Thus, if $d = 4$, then $\mu(P) = (d-1)^2 - \lfloor \frac{d}{2} \rfloor = 7$ if and only if either $C_4$ is a Płoski curve and $P$ is its singular point or $C_4$ has singularity $\mathbb{E}_7$ at the point $P$ (see Example 1.6). For $d \geqslant 5$, Płoski proved

**Theorem 1.8** ([10, Theorem 1.4])**.** If $d \geqslant 5$, then $\mu(P) = (d-1)^2 - \lfloor \frac{d}{2} \rfloor$ if and only if $C_d$ is a Płoski curve and $P$ is its singular point.

This result gives a *very good* answer to Question 1.4. The main goal of this paper is to give an answer to Question 1.4. using log canonical thresholds. Namely, we will prove that

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) \geqslant \frac{2d-3}{(d-1)^2}$$

provided that $m_P < d$, and we will describe $C_d$ in the case when $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{2d-3}{(d-1)^2}$. To present this description, we need

**Definition 1.9.** The curve $C_d$ has singularity of type $\mathbb{T}_r$ (resp., $\mathbb{K}_r$, $\widetilde{\mathbb{T}}_r$, $\widetilde{\mathbb{K}}_r$) at the point $P$ if the curve $C_d$ can be given by $x_1^r = x_1 x_2^r$ (resp., $x_1^r = x_2^{r+1}$, $x_2 x_1^{r-1} = x_1 x_2^r$, $x_2 x_1^{r-1} = x_2^{r+1}$) up to analytic change of coordinates at the point $P$.

Note that $\mathbb{T}_2 = \mathbb{A}_3$, $\mathbb{K}_2 = \mathbb{A}_2$, $\widetilde{\mathbb{T}}_2 = \widetilde{\mathbb{K}}_2 = \mathbb{A}_1$, $\widetilde{\mathbb{K}}_3 = \mathbb{D}_5$, $\widetilde{\mathbb{T}}_3 = \mathbb{D}_6$, $\mathbb{K}_3 = \mathbb{E}_6$ and $\mathbb{T}_3 = \mathbb{E}_7$. Furthermore, since we assume that $d \geqslant 3$, the formula in Example 1.2 gives

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \begin{cases} \dfrac{2d-3}{(d-1)^2} \text{ if } C_d \text{ has } \mathbb{T}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-1}{d(d-1)} \text{ if } C_d \text{ has } \mathbb{K}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-5}{d^2-3d+1} \text{ if } C_d \text{ has } \widetilde{\mathbb{T}}_{d-1} \text{ singularity at } P, \\ \dfrac{2d-3}{d(d-2)} \text{ if } C \text{ has } \widetilde{\mathbb{K}}_{d-1} \text{ singularity at } P, \end{cases}$$

where $\frac{2}{d} < \frac{2d-3}{(d-1)^2} < \frac{2d-1}{d(d-1)} < \frac{2d-5}{d^2-3d+1} \leqslant \frac{2d-3}{d(d-2)}$. In this paper we will prove

**Theorem 1.10.** Suppose that $d \geqslant 4$ and $\mathrm{lct}_P(\mathbb{P}^2, C_d) \leqslant \frac{2d-3}{d(d-2)}$. Then one of the following holds:

(1) $m_P = d$,
(2) the curve $C_d$ has singularity of type $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point $P$,
(3) $d = 4$ and $C_d$ is a Płoski quartic curve (in this case $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{5}{8}$).

This result describes the *five worst* singularities that $C_d$ can have at the point $P$. In particular, Theorem 1.10 answers Question 1.4. This answer is very different from the answer given by Theorem 1.8. Indeed, if $C_d$ is a Płoski curve and $P$ is its singular point, then the formula in Example 1.2 gives

$$\mathrm{lct}_P\big(\mathbb{P}^2, C_d\big) = \frac{5}{2d} > \frac{2d-3}{(d-1)^2}.$$

The proof of Theorem 1.10 implies one result that is interesting on its own. To describe it, let us identify the curve $C_d$ with a point in the space $|\mathcal{O}_{\mathbb{P}^2}(d)|$ that parameterizes all (not necessarily reduced) plane curves of degree $d$. Since the group $\mathrm{PGL}_3(\mathbb{C})$ acts on $|\mathcal{O}_{\mathbb{P}^2}(d)|$, it is natural to ask whether $C_d$ is GIT-stable (resp., GIT-semistable) for this action or not. For small $d$, its answer is classical and immediately follows from the Hilbert–Mumford criterion (see [9, Chapter 2.1]).

**Example 1.11** ([9, Chapter 4.2])**.** If $d = 3$, then $C_3$ is GIT-stable (resp., GIT-semistable) if and only if $C_3$ is smooth (resp., $C_3$ has at most $\mathbb{A}_1$ singularities). If $d = 4$, then $C_4$ is GIT-stable (resp., GIT-semistable) if and only if $C_4$ has at most $\mathbb{A}_1$ and $\mathbb{A}_2$ singularities (resp., $C_4$ has at most singular double points and $C_4$ is not a union of a cubic with an inflectional tangent line).

Paul Hacking, Hosung Kim and Yongnam Lee noticed that the log canonical threshold $\mathrm{lct}(\mathbb{P}^2, C_d)$ and GIT-stability of the curve $C_d$ are closely related. In particular, they proved

**Theorem 1.12** ([5, Propositions 10.2 and 10.4], [7, Theorem 2.3])**.** If $\mathrm{lct}(\mathbb{P}^2, C_d) \geqslant \frac{3}{d}$, then the curve $C_d$ is GIT-semistable. If $d \geqslant 4$ and $\mathrm{lct}(\mathbb{P}^2, C_d) > \frac{3}{d}$, then the curve $C_d$ is GIT-stable.

This gives a *sufficient* condition for the curve $C_d$ to be GIT-stable (resp, GIT-semistable). However, this condition is not a *necessary* condition. Let us give two examples that illustrate this.

**Example 1.13** ([13, p. 268], [5, Example 10.5])**.** Suppose that $d = 5$, the quintic curve $C_5$ is given by

$$x^5 + \Big(y^2 - xz\Big)^2\Big(\frac{x}{4} + y + z\Big) = x^2\Big(y^2 - xz\Big)\Big(x + 2y\Big),$$

and $P = [0 : 0 : 1]$. Then $C_5$ is irreducible and has singularity $\mathbb{A}_{12}$ at the point $P$. In particular, it is rational. Furthermore, the curve $C_5$ is GIT-stable (see, for example, [9, Chapter 4.2]). On the other hand, it follows from Example 1.2 that

$$\mathrm{lct}\Big(\mathbb{P}^2, C_5\Big) = \mathrm{lct}_P\Big(\mathbb{P}^2, C_5\Big) = \frac{1}{2} + \frac{1}{13} = \frac{15}{26} < \frac{3}{5}.$$

**Example 1.14.** Suppose that $C_d$ is a Płoski curve. Let $P$ be its singular point, and let $L$ be a general line in $\mathbb{P}^2$. Then

$$\mathrm{lct}\Big(\mathbb{P}^2, C_d + L\Big) = \mathrm{lct}\Big(\mathbb{P}^2, C_d\Big) = \mathrm{lct}_P\Big(\mathbb{P}^2, C_d\Big) = \frac{5}{2d} < \frac{3}{d}.$$

On the other hand, if $d$ is even, then $C_d$ is GIT-semistable, and $C_d + L$ is GIT-stable. This follows from the Hilbert–Mumford criterion. Similarly, if $d$ is odd, then $C_d$ is GIT-unstable, and $C_d + L$ is GIT-semistable.

In this paper we will prove the following result that complements Theorem 1.12.

**Theorem 1.15.** If $\mathrm{lct}(\mathbb{P}^2, C_d) < \frac{5}{2d}$, then $C_d$ is GIT-unstable. Moreover, if $\mathrm{lct}(\mathbb{P}^2, C_d) \leqslant \frac{5}{2d}$, then $C_d$ is not GIT-stable. Furthermore, if $\mathrm{lct}(\mathbb{P}^2, C_d) = \frac{5}{2d}$, then $C_d$ is GIT-semistable if and only if $C_d$ is an even Płoski curve.

Example 1.14 shows that this result is *sharp*. Surprisingly, its proof is very similar to the proof of Theorem 1.10. In fact, we will give a combined proof of both these theorems in Section 3.

In this paper we will also prove one application of Theorem 1.10. To describe it, we need

**Definition 1.16** ([12, Appendix A], [3, Definition 1.20])**.** For a given smooth variety $V$ equipped with an ample $\mathbb{Q}$-divisor $H_V$, let $\alpha_V^{H_V} : V \to \mathbb{R}_{\geqslant 0}$ be a function defined as

$$\alpha_V^{H_V}(O) = \sup\left\{\lambda \in \mathbb{Q}\ \left|\ \begin{array}{l}\text{the pair } (V, \lambda D_V) \text{ is log canonical at } O\\ \text{for every effective } \mathbb{Q}\text{-divisor } D_V \sim_{\mathbb{Q}} H_V\end{array}\right.\right\}.$$

Denote its infimum by $\alpha(V, H_V)$.

Let $S_d$ be a smooth surface in $\mathbb{P}^3$ of degree $d \geqslant 3$, let $H_{S_d}$ be its hyperplane section, let $O$ be a point in $S_d$, and let $T_O$ be the hyperplane section of $S_d$ that is singular at $O$. Similar to $\mathrm{lct}_P(\mathbb{P}^2, C_d)$, we can define

$$\mathrm{lct}_O\Big(S_d, T_O\Big) = \sup\Big\{\lambda \in \mathbb{Q}\ \Big|\ \text{the log pair } \Big(S_d, \lambda T_O\Big) \text{ is log canonical at } O\Big\}.$$

Then $\alpha_{S_d}^{H_{S_d}}(O) \leqslant \mathrm{lct}_O(S_d, T_O)$ by Definition 1.16. Note that $T_O$ is reduced, since the surface $S_d$ is smooth. In this paper we prove

**Theorem 1.17.** If $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$, then

$$\alpha_{S_d}^{H_{S_d}}(O) = \mathrm{lct}_O\big(S_d, T_O\big) \in \left\{\frac{2}{d}, \frac{2d-3}{(d-1)^2}, \frac{2d-1}{d(d-1)}, \frac{2d-5}{d^2-3d+1}\right\}.$$

Similarly, if $\alpha(S_d, H_{S_d}) < \frac{2d-3}{d(d-2)}$, then

$$\alpha\big(S_d, H_{S_d}\big) = \inf_{O\in S_d}\Big\{\mathrm{lct}_O\big(S_d, T_O\big)\Big\} \in \left\{\frac{2}{d}, \frac{2d-3}{(d-1)^2}, \frac{2d-1}{d(d-1)}, \frac{2d-5}{d^2-3d+1}\right\}.$$

If $d = 3$, then we can drop the condition $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$ in Theorem 1.17, since $\frac{2d-3}{d(d-2)} = 1$ in this case. Thus, Theorem 1.17 implies

**Corollary 1.18** ([3, Corollary 1.24])**.** Suppose that $d = 3$. Then $\alpha_{S_3}^{H_{S_3}}(O) = \mathrm{lct}_O(S_3, T_O)$.

If $d \geqslant 4$, we cannot drop the condition $\alpha_{S_d}^{H_{S_d}}(O) < \frac{2d-3}{d(d-2)}$ in Theorem 1.17 in general. Let us give two examples that illustrate this.

**Example 1.19.** Suppose that $d = 4$. Let $S_4$ be a quartic surface in $\mathbb{P}^3$ that is given by

$$t^3x + t^2yz + xyz(y+z) = 0,$$

and let $O$ be the point $[0 : 0 : 0 : 1]$. Then $S_4$ is smooth, and $T_O$ has singularity $\mathbb{A}_1$ at $O$, which implies that $\mathrm{lct}_O(S_4, T_O) = 1$. Let $L_y$ be the line $x = y = 0$, let $L_z$ be the line $x = z = 0$, and let $C_2$ be the conic $y + z = xt + yz = 0$. Then $L_y$, $L_z$ and $C_2$ are contained in $S_4$, and $O = L_y \cap L_z \cap C_2$. Moreover,

$$L_y + L_z + \frac{1}{2}C_2 \sim 2H_{S_4},$$

because the divisor $2L_y + 2L_z + C_2$ is cut out on $S_4$ by $tx + yz = 0$. Furthermore, the log pair $(S_4, L_y + L_z + \frac{1}{2}C_2)$ is not log canonical at $O$, so that $\alpha_{S_4}^{H_{S_4}}(O) < 1$ by Definition 1.16.

**Example 1.20.** Suppose that $d \geqslant 5$ and $T_O$ has $\mathbb{A}_1$ singularity at $O$. Then $\mathrm{lct}_O(S_d, T_O) = 1$. Let $f\colon \widetilde{S}_d \to S_d$ be a blow up of the point $O$. Denote by $E$ its exceptional curve. Then

$$\Big(f^*(H_{S_d}) - \frac{11}{5}E\Big)^2 = 5 - \frac{121}{25} > 0.$$

Hence, it follows from Riemann–Roch theorem there is an integer $n \geqslant 1$ such that the linear system $|f^*(5nH_{S_d}) - 11nE|$ is not empty. Pick a divisor $\widetilde{D}$ in this linear system, and denote by $D$ its image on $S_d$. Then $(S_d, \frac{1}{5n}D)$ is not log canonical at $P$, since $\mathrm{mult}_P(D) \geqslant 11n$. On the other hand, $\frac{1}{5n}D \sim_{\mathbb{Q}} H_{S_d}$ by construction, so that $\alpha_{S_d}^{H_d}(O) < 1$ by Definition 1.16.

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## 2. Preliminaries

In this section, we present results that will be used in the proof of Theorems 1.10, 1.15, 1.17. Let $S$ be a smooth surface, let $D$ be an effective non-zero $\mathbb{Q}$-divisor on the surface $S$, and let $P$ be a point in the surface $S$. Write

$$D = \sum_{i=1}^{r} a_iC_i,$$

where each $C_i$ is an irreducible curve on the surface $S$, and each $a_i$ is a non-negative rational number. Let us recall

**Definition 2.1** ([4, § 6])**.** Let $\pi\colon \widetilde{S}\to S$ be a birational morphism such that $\widetilde{S}$ is smooth. Then $\pi$ is a composition of blow ups of smooth points. For each $C_i$, denote by $\widetilde{C}_i$ its proper transform on the surface $\widetilde{S}$. Let $F_1,\ldots,F_n$ be $\pi$-exceptional curves. Then

$$K_{\widetilde{S}}+\sum_{i=1}^{r}a_i\widetilde{C}_i+\sum_{j=1}^{n}b_jF_j\sim_{\mathbb{Q}}\pi^*\big(K_S+D\big)$$

for some rational numbers $b_1,\ldots,b_n$. Suppose, in addition, that $\sum_{i=1}^{r}\widetilde{C}_i+\sum_{j=1}^{n}F_j$ is a divisor with simple normal crossings. Then the log pair $(S,D)$ is said to be *log canonical* at $P$ if and only if the following two conditions are satisfied:

- $a_i\leqslant 1$ for every $C_i$ such that $P\in C_i$,
- $b_j\leqslant 1$ for every $F_j$ such that $\pi(F_j)=P$.

Similarly, the log pair $(S,D)$ is said to be *Kawamata log terminal* at $P$ if and only if $a_i<1$ for every $C_i$ such that $P\in C_i$, and $b_j<1$ for every $F_j$ such that $\pi(F_j)=P$.

Using just this definition, one can easily prove

**Lemma 2.2.** Suppose that $r=3$, $P\in C_1\cap C_2\cap C_3$, the curves $C_1$, $C_2$ and $C_3$ are smooth at $P$, $a_1<1$, $a_2<1$ and $a_3<1$. Moreover, suppose that both curves $C_1$ and $C_2$ intersect the curve $C_3$ transversally at $P$. Furthermore, suppose that $(S,D)$ is not Kawamata log terminal at $P$. Put $k=\mathrm{mult}_P(C_1\cdot C_2)$. Then $k(a_1+a_2)+a_3\geqslant k+1$.

*Proof.* Put $S_0=S$ and consider a sequence of blow ups

$$S_k\xrightarrow{\pi_k}S_{k-1}\xrightarrow{\pi_{k-1}}\cdots\xrightarrow{\pi_3}S_2\xrightarrow{\pi_2}S_1\xrightarrow{\pi_1}S_0,$$

where each $\pi_j$ is the blow up of the intersection point of the proper transforms of the curves $C_1$ and $C_2$ on the surface $S_{j-1}$ that dominates $P$ (such point exists, since $k=\mathrm{mult}_P(C_1\cdot C_2)$). For each $\pi_j$, denote by $E_j^k$ the proper transform of its exceptional curve on $S_k$. For each $C_i$, denote by $C_i^k$ its proper transform on the surface $S_k$. Then

$$K_{S_k}+\sum_{i=1}^{n}a_iC_i^k+\sum_{j=1}^{k}\Big(j\big(a_1+a_2\big)+a_3-j\Big)E_j^k\sim_{\mathbb{Q}}(\pi_1\circ\pi_2\circ\cdots\circ\pi_k)^*\Big(K_S+D\Big),$$

and $\sum_{i=1}^{n}C_i^k+\sum_{j=1}^{k}E_j$ is a simple normal crossing divisor in every point of $\cup_{j=1}^{k}E_j$. Thus, it follows from Definition 2.1 that there exists $l\in\{1,\ldots,k\}$ such that $l(a_1+a_2)+a_3\geqslant l+1$, because $(S,D)$ is not Kawamata log terminal at $P$. If $l=k$, then we are done. So, we may assume that $l<k$. If $k(a_1+a_2)+a_3<k+1$, then $a_1+a_2<1+\frac{1}{k}-a_3\frac{1}{k}$, which implies that

$$l+1\leqslant l\big(a_1+a_2\big)+a_3<\left(l+\frac{l}{k}-a_3\frac{l}{k}\right)+a_3=l+\frac{l}{k}+a_3\left(1-\frac{l}{k}\right)\leqslant l+\frac{l}{k}+\left(1-\frac{l}{k}\right)=l+1,$$

because $a_3<1$. Thus, the obtained contradiction shows that $k(a_1+a_2)+a_3\geqslant k+1$. $\square$

**Corollary 2.3.** Suppose that $r=2$, $P\in C_1\cap C_2$, the curves $C_1$ and $C_2$ are smooth at $P$, $a_1<1$ and $a_2<1$. Put $k=\mathrm{mult}_P(C_1\cdot C_2)$. If $(S,D)$ is not Kawamata log terminal at $P$, then $k(a_1+a_2)\geqslant k+1$.

The log pair $(S,D)$ is called *log canonical* if it is log canonical at every point of $S$. Similarly, the log pair $(S,D)$ is called *Kawamata log terminal* if it is Kawamata log terminal at every point of the surface $S$.

*Remark* 2.4. Let $R$ be any effective $\mathbb{Q}$-divisor on $S$ such that $R\sim_{\mathbb{Q}}D$ and $R\neq D$. Put

$$D_\epsilon=(1+\epsilon)D-\epsilon R,$$

where $\epsilon$ is a non-negative rational number. Then $D_\epsilon\sim_{\mathbb{Q}}D$. Moreover, since $R\neq D$, there exists the greatest rational number $\epsilon_0\geqslant 0$ such that the divisor $D_{\epsilon_0}$ is effective. Then $\mathrm{Supp}(D_{\epsilon_0})$

does not contain at least one irreducible component of $\mathrm{Supp}(R)$. Moreover, if $(S, D)$ is not log canonical at $P$, and $(S, R)$ is log canonical at $P$, then $(S, D_{\epsilon_0})$ is not log canonical at $P$ by Definition 2.1, because

$$D = \frac{1}{1+\epsilon_0} D_{\epsilon_0} + \frac{\epsilon_0}{1+\epsilon_0} R$$

and $\frac{1}{1+\epsilon_0} + \frac{\epsilon_0}{1+\epsilon_0} = 1$. Similarly, if the log pair $(S, D)$ is not Kawamata log terminal at $P$, and $(S, R)$ is Kawamata log terminal at $P$, then $(S, D_{\epsilon_0})$ is not Kawamata log terminal at $P$.

The following result is well-known.

**Lemma 2.5** ([4, Exercise 6.18])**.** If $(S, D)$ is not log canonical at $P$, then $\mathrm{mult}_P(D) > 1$. Similarly, if $(S, D)$ is not Kawamata log terminal at $P$, then $\mathrm{mult}_P(D) \geqslant 1$.

Combining with

**Lemma 2.6** ([4, Lemma 5.36])**.** Suppose that $S$ is a smooth surface in $\mathbb{P}^3$, and $D \sim_{\mathbb{Q}} H_S$, where $H_S$ is a hyperplane section of $S$. Then each $a_i$ does not exceed 1.

Lemma 2.5 gives

**Corollary 2.7.** Suppose that $S$ is a smooth surface in $\mathbb{P}^3$, and $D \sim_{\mathbb{Q}} H_S$, where $H_S$ is a hyperplane section of $S$. Then $(S, D)$ is log canonical outside of finitely many points.

The following result is a special case of a much more general result, which is known as Shokurov's connectedness principle (see, for example, [4, Theorem 6.3.2]).

**Lemma 2.8** ([11, Theorem 6.9])**.** If $-(K_S + D)$ is big and nef, then the locus where $(S, D)$ is not Kawamata log terminal is connected.

**Corollary 2.9.** Let $C_d$ be a reduced curve in $\mathbb{P}^2$ of degree $d$, let $O$ and $Q$ be two points in $C_d$ such that $O \neq Q$. If $\mathrm{lct}_O(\mathbb{P}^2, C_d) < \frac{3}{d}$, then $\mathrm{lct}_Q(\mathbb{P}^2, C_d) \geqslant \frac{3}{d}$.

Let $\pi_1 \colon S_1 \to S$ be a blow up of the point $P$, and let $E_1$ be the $\pi_1$-exceptional curve. Denote by $D^1$ the proper transform of the divisor $D$ on the surface $S_1$ via $\pi_1$. Then the log pair $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is often called *the log pull back* of the log pair $(S, D)$, because

$$K_{S_1} + D^1 + \Big(\mathrm{mult}_P(D) - 1\Big)E_1 \sim_{\mathbb{Q}} \pi_1^*\big(K_S + D\big).$$

This $\mathbb{Q}$-rational equivalence implies that the log pair $(S, D)$ is not log canonical at $P$ provided that $\mathrm{mult}_P(D) > 2$. Similarly, if $\mathrm{mult}_P(D) \geqslant 2$, then the singularities of the log pair $(S, D)$ are not Kawamata log terminal at the point $P$.

*Remark* 2.10. The log pair $(S, D)$ is log canonical at $P$ if and only if $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is log canonical at every point of the curve $E_1$. Similarly, the log pair $(S, D)$ is Kawamata log terminal at $P$ if and only if $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is Kawamata log terminal at every point of the curve $E_1$.

Let $Z$ be an irreducible curve on $S$ that contains $P$. Suppose that $Z$ is smooth at $P$, and $Z$ is not contained in $\mathrm{Supp}(D)$. Let $\mu$ be a non-negative rational number. The following result is a very special case of a much more general result known as *Inversion of Adjunction* (see, for example, [11, § 3.4] or [4, Theorem 6.29]).

**Theorem 2.11** ([11, Corollary 3.12], [4, Exercise 6.31], [2, Theorem 7])**.** Suppose that the log pair $(S, \mu Z + D)$ is not log canonical at $P$ and $\mu \leqslant 1$. Then $\mathrm{mult}_P(D \cdot Z) > 1$.

This result implies

**Theorem 2.12.** Suppose that $(S, \mu Z + D)$ is not Kawamata log terminal at $P$, and $\mu < 1$. Then $\mathrm{mult}_P(D \cdot Z) > 1$.

*Proof.* The log pair $(S, Z + D)$ is not log canonical at $P$, because $\mu < 1$, and $(S, \mu Z + D)$ is not Kawamata log terminal at $P$. Then $\mathrm{mult}_P(D \cdot Z) > 1$ by Theorem 2.11. $\square$

Theorems 2.11 and 2.12 imply

**Lemma 2.13.** If $(S, D)$ is not log canonical at $P$ and $\mathrm{mult}_P(D) \leqslant 2$, then there exists a *unique* point in $E_1$ such that $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not log canonical at it. Similarly, if $(S, D)$ is not Kawamata log terminal at $P$, and $\mathrm{mult}_P(D) < 2$, then there exists a *unique* point in $E_1$ such that $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not Kawamata log terminal at it.

*Proof.* If $\mathrm{mult}_P(D) \leqslant 2$ and $(S_1, D^1 + (\lambda \mathrm{mult}_P(D) - 1)E_1)$ is not log canonical at two distinct points $P_1$ and $\widetilde{P}_1$, then

$$2 \geqslant \mathrm{mult}_P\big(D\big) = D^1 \cdot E_1 \geqslant \mathrm{mult}_{P_1}\Big(D^1 \cdot E_1\Big) + \mathrm{mult}_{\widetilde{P}_1}\Big(D^1 \cdot E_1\Big) > 2$$

by Theorem 2.11. By Remark 2.10, this proves the first assertion. Similarly, we can prove the second assertion using Theorem 2.12 instead of Theorem 2.11. ☐

The following result can be proved similarly to the proof of Lemma 2.5. Let us show how to prove it using Theorem 2.12.

**Lemma 2.14.** Suppose that $(S, D)$ is not Kawamata log terminal at $P$, and $(S, D)$ is Kawamata log terminal in a punctured neighborhood of the point $P$, then $\mathrm{mult}_P(D) > 1$.

*Proof.* By Remark 2.10, the log pair $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$. Moreover, if $\mathrm{mult}_P(D) < 2$, then $(S_1, D^1 + (\mathrm{mult}_P(D) - 1)E_1)$ is Kawamata log terminal at a punctured neighborhood of the point $P_1$. Thus, if $\mathrm{mult}_P(D) \leqslant 1$, then $\mathrm{mult}_P\big(D\big) = D^1 \cdot E_1 > 1$ by Theorem 2.12, which is absurd. ☐

Let $Z_1$ and $Z_2$ be two irreducible curves on the surface $S$ such that $Z_1$ and $Z_2$ are not contained in $\mathrm{Supp}(D)$. Suppose that $P \in Z_1 \cap Z_2$, the curves $Z_1$ and $Z_2$ are smooth at $P$, the curves $Z_1$ and $Z_2$ intersect each other transversally at $P$. Let $\mu_1$ and $\mu_2$ be non-negative rational numbers.

**Theorem 2.15** ([2, Theorem 13])**.** Suppose that the log pair $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ is not log canonical at the point $P$, and $\mathrm{mult}_P(D) \leqslant 1$. Then either $\mathrm{mult}_P(D \cdot Z_1) > 2(1 - \mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) > 2(1 - \mu_1)$ (or both).

This result implies

**Theorem 2.16.** Suppose that $(S, \mu_1 Z_1 + \mu_2 Z_2 + D)$ is not Kawamata log terminal at $P$, and $\mathrm{mult}_P(D) < 1$. Then either $\mathrm{mult}_P(D \cdot Z_1) \geqslant 2(1 - \mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) \geqslant 2(1 - \mu_1)$ (or both).

*Proof.* Let $\lambda$ be a rational number such that

$$\frac{1}{\mathrm{mult}_P(D)} \geqslant \lambda > 1.$$

Then $(S, D + \lambda\mu_1 Z_1 + \lambda\mu_2 Z_2)$ is not log canonical at $P$. Now it follows from Theorem 2.15 that either $\mathrm{mult}_P(D \cdot Z_1) > 2(1 - \lambda\mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) > 2(1 - \lambda\mu_1)$ (or both). Since we can choose $\lambda$ to be as close to 1 as we wish, this implies that either $\mathrm{mult}_P(D \cdot Z_1) \geqslant 2(1 - \mu_2)$ or $\mathrm{mult}_P(D \cdot Z_2) \geqslant 2(1 - \mu_1)$ (or both). ☐

## 3. Reduced plane curves

The purpose of this section is to prove Theorems 1.10 and 1.15. Let $C_d$ be a *reduced* plane curve in $\mathbb{P}^2$ of degree $d \geqslant 4$, and let $P$ be a point in $C_d$. Put $\lambda_1 = \frac{2d-3}{d(d-2)}$ and $\lambda_2 = \frac{5}{2d}$. To prove Theorem 1.10, we have to show that if the log pair $(\mathbb{P}^2, \lambda_1 C_d)$ is not Kawamata log terminal at the point $P$, then one of the following assertions hold:

- $\mathrm{mult}_P(C_d) = d$,
- $C_d$ has singularity $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point $P$,
- $d = 4$ and $C_4$ is a Płoski curve (see Definition 1.7).

To prove Theorem 1.15, we have to show that if $(\mathbb{P}^2, \lambda_2 C_d)$ is not Kawamata log terminal, then either $C_d$ is GIT-unstable or $C_d$ is an even Płoski curve. In the rest of the section, we will do this simultaneously. Let us start with few preliminary results.

**Lemma 3.1.** The following inequalities hold:

(i) $\lambda_1 < \frac{2}{d-1}$,

(ii) $\lambda_1 < \frac{2k+1}{kd}$ for every positive integer $k \leqslant d-3$,

(iii) if $d \geqslant 5$, then $\lambda_1 < \frac{2k+1}{kd+1}$ for every positive integer $k \leqslant d-4$,

(iv) $\lambda_1 < \frac{3}{d}$,

(v) $\lambda_1 < \frac{2}{d-2}$,

(vi) $\lambda_1 < \frac{6}{3d-4}$,

(vii) if $d \geqslant 5$, then $\lambda_1 < \lambda_2$.

*Proof.* The equality $\frac{2}{d-1} = \lambda_1 + \frac{d-3}{d(d-1)(d-2)}$ implies (i). Let $k$ be positive integer. If $k = d-2$, then $\lambda_1 = \frac{2k+1}{kd}$. This implies (ii), because $\frac{2k+1}{kd} = \frac{2}{d} + \frac{1}{kd}$ is a decreasing function on $k$ for $k \geqslant 1$. Similarly, if $k = d-4$ and $d \geqslant 4$, then $\lambda_1 = \frac{2k+1}{kd+1} - \frac{3}{d(d-2)(d^2-4d+1)} < \frac{2k+1}{kd+1}$. This implies (iii), since $\frac{2k+1}{kd+1} = \frac{2}{d} + \frac{d-2}{d(kd+1)}$ is a decreasing function on $k$ for $k \geqslant 1$. The equality $\lambda_1 = \frac{3}{d} - \frac{d-3}{d(d-2)}$ proves (iv). Note that (v) follows from (i). Since $\frac{6}{3d-4} > \frac{2}{d-1}$, (vi) also follows from (i). Finally, the equality $\lambda_1 = \lambda_2 - \frac{d-4}{2d(d-2)}$ implies (vii). $\square$

We may assume that $P = [0:0:1]$. Then $C_d$ is given by $F_d(x,y,z) = 0$, where $F_d(x,y,z)$ is a homogeneous polynomial of degree $d$. Put $x_1 = \frac{x}{z}$, $x_2 = \frac{y}{z}$ and $f_d(x_1,x_2) = F_d(x_1,x_2,1)$. Then

$$f_d(x_1,x_2) = \sum_{\substack{i\geqslant 0, j\geqslant 0,\\ m_0\leqslant i+j\leqslant d}} \epsilon_{ij} x_1^i x_2^j,$$

where each $\epsilon_{ij}$ is a complex number. For every positive integers $a$ and $b$, define the weight of the polynomial $f_d(x_1,x_2)$ as

$$\mathrm{wt}_{(a,b)}\big(f_d(x_1,x_2)\big) = \min\Big\{ai + bj \ \Big|\ \epsilon_{ij} \neq 0\Big\}.$$

Then the Hilbert–Mumford criterion implies

**Lemma 3.2** ([7, Lemma 2.1])**.** Let $a$ and $b$ be positive integers. If $C_d$ is GIT-stable, then

$$\mathrm{wt}_{(a,b)}\Big(f_d\big(x_1,x_2\big)\Big) < \frac{d}{3}\big(a+b\big).$$

Similarly, if $C_d$ is GIT-semistable, then $\mathrm{wt}_{(a,b)}(f_d(x_1,x_2)) \leqslant \frac{d}{3}(a+b)$.

Let $f_1\colon S_1 \to \mathbb{P}^2$ be a blow up of the point $P$. Denote by $E_1$ the exceptional curve of the blow up $f_1$. Denote by $C_d^1$ the proper transform on $S_1$ of the curve $C_d$.

**Lemma 3.3.** If $\mathrm{mult}_P(C_d) > \frac{2d}{3}$, then $C_d$ is GIT-unstable. Let $O$ be a point in $E_1$. If

$$\mathrm{mult}_P(C_d) + \mathrm{mult}_O(C_d^1) > d,$$

then $C_d$ is GIT-unstable.

*Proof.* Since $\mathrm{mult}_P(C_d) = \mathrm{wt}_{(1,1)}(f_d(x_1,x_2))$, the first assertion follows from Lemma 3.2. Let us prove the second assertion. We may assume that $O$ is contained in the proper transform of the line in $\mathbb{P}^2$ that is given by $x = 0$. Then

$$\mathrm{wt}_{(2,1)}\big(f_d(x_1,x_2)\big) = \mathrm{mult}_P(C_d) + \mathrm{mult}_O(C_d^1),$$

so that the second assertion also follows from Lemma 3.2. $\square$

Now we are ready to prove Theorems 1.10 and 1.15. To do this, we may assume that $C_d$ is not a union of $d$ lines passing through the point $P$. Suppose, in addition, that

(**A**) either $(\mathbb{P}^2, \lambda_1 C_d)$ is not Kawamata log terminal at $P$,
(**B**) or $(\mathbb{P}^2, \lambda_2 C_d)$ is not Kawamata log terminal at $P$.

We will show that (**A**) implies that either $C_d$ has singularity $\mathbb{T}_{d-1}$, $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{T}}_{d-1}$ or $\widetilde{\mathbb{K}}_{d-1}$ at the point $P$, or $C_d$ is a Płoski quartic curve. Similarly, we will show that (**B**) implies that either $C_d$ is GIT-unstable (i.e. $C_d$ is not GIT-semistable), or $C_d$ is an even Płoski curve. If (**A**) holds, let $\lambda = \lambda_1$. If (**B**) holds, let $\lambda = \lambda_2$.

If $d = 4$, then $\lambda_1 = \lambda_2$. If $d \geqslant 5$, then $\lambda_1 < \lambda_2$ by Lemma 3.1(vii). Since $C_d$ is reduced and $\lambda < 1$, the log pair $(\mathbb{P}^2, \lambda C_d)$ is Kawamata log terminal outside of finitely many points. Thus, it is Kawamata log terminal outside of $P$ by Lemma 2.8.

Put $m_0 = \mathrm{mult}_P(C_d)$. Then the log pair $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal at some point $P_1 \in E_1$ by Remark 2.10. Note that we have

$$K_{S_1} + \lambda C_d^1 + \Big(\lambda m_0 - 1\Big)E_1 \sim_{\mathbb{Q}} f_1^*\Big(K_{\mathbb{P}^2} + \lambda C_d\Big).$$

Let $f_2\colon S_2 \to S_1$ be a blow up of the point $P_1$, and let $E_2$ be its exceptional curve. Denote by $C_d^2$ the proper transform on $S_2$ of the curve $C_d$, and denote by $E_1^2$ the proper transform on $S_2$ of the curve $E_1$. Put $m_1 = \mathrm{mult}_{P_1}(C_d^1)$. Then

$$K_{S_2} + \lambda C_d^2 + \big(\lambda m_0 - 1\big)E_1^2 + \big(\lambda(m_0 + m_1) - 2\big)E_2 \sim_{\mathbb{Q}} f_2^*\Big(K_{S_1} + \lambda C_d^1 + \big(\lambda m_0 - 1\big)E_1\Big).$$

By Remark 2.10, the log pair $(S_2, \lambda C_d^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not Kawamata log terminal at some point $P_2 \in E_2$. Let $f_3\colon S_3 \to S_2$ be a blow up of this point, and let $E_3$ be the $f_3$-exceptional curve. Denote by $C_d^3$ the proper transform on $S_3$ of the curve $C_d$, denote by $E_1^3$ the proper transform on $S_3$ of the curve $E_1$, and denote by $E_2^3$ the proper transform on $S_3$ of the curve $E_2$. Put $m_2 = \mathrm{mult}_{P_2}(C_d^2)$. Then

$$\begin{aligned} K_{S_3} + \lambda_2 C_d^3 + \big(\lambda_2 m_0 - 1\big)E_1^3 + \\ + \big(\lambda_2(m_0 + m_1) - 2\big)E_2^3 + \big(\lambda_2(2m_0 + m_1 + m_2) - 4\big)E_3 \sim_{\mathbb{Q}} \\ \sim_{\mathbb{Q}} f_3^*\Big(K_{S_2} + \lambda_2 C_d^2 + \big(\lambda_2 m_0 - 1\big)E_1^2 + \big(\lambda_2(m_0 + m_1) - 2\big)E_2\Big). \end{aligned}$$

Thus, the log pair $(S_3, \lambda_2 C_d^3 + (\lambda_2 m_0 - 1)E_1^3 + (\lambda_2(m_0 + m_1) - 2)E_2^3 + (\lambda_2(2m_0 + m_1 + m_2) - 4)E_3)$ is not Kawamata log terminal at some point $P_3 \in E_3$ by Remark 2.10. Note that the divisor $\lambda_2 C_d^3 + (\lambda_2 m_0 - 1)E_1^3 + (\lambda_2(m_0 + m_1) - 2)E_2^3 + (\lambda_2(2m_0 + m_1 + m_2) - 4)E_3$ is effective by Lemma 2.5.

**Lemma 3.4.** One has $\lambda m_0 < 2$.

*Proof.* Since $C_d$ is not a union of $d$ lines passing through $P$, we have $m_0 \leqslant d - 1$. Thus, if (**A**) holds, then $\lambda m_0 < 2$ by Lemma 3.1(i), because $d \geqslant 4$. Similarly, if (**B**) holds, then $m_0 \leqslant \frac{2d}{3}$ by Lemma 3.3, which implies that $\lambda m_0 \leqslant \frac{10}{6} < 2$. □

Thus, the log pair $(S_1, \lambda C_d^1 + (\lambda m_0 - 1)E_1)$ is Kawamata log terminal outside of $P_1$ by Lemma 2.13. Note that $P_1 \in C_d^1$, because the log pair $(S_1, (\lambda m_0 - 1)E_1)$ is not Kawamata log terminal at $P_1$. Thus, we have $m_1 > 0$.

Let $L$ be the line in $\mathbb{P}^2$ whose proper transform on $S_1$ contains the point $P_1$. Such a line exists and it is unique. By a suitable linear change of coordinates, we may assume that $L$ is given by $x = 0$. Denote by $L^1$ the proper transform of the line $L$ on the surface $S_1$.

**Lemma 3.5.** Suppose that (**A**) holds and $m_0 = d - 1$. Then $C_d$ has singularity $\mathbb{K}_{d-1}$, $\widetilde{\mathbb{K}}_{d-1}$, $\mathbb{T}_{d-1}$ or $\widetilde{\mathbb{T}}_{d-1}$ at the point $P$.

*Proof.* Suppose that $L$ is not an irreducible component of the curve $C_d$. Then $m_0 + m_1 \leqslant d$, because

$$d - 1 - m_0 = C_d^1 \cdot L^1 \geqslant m_1.$$

Since $m_0 = d-1$, this gives $m_1 = 1$. Then $P_1 \in C_d^1$ and the curve $C_d^1$ is smooth at $P_1$. Put $k = \mathrm{mult}_{P_1}(C_d^1 \cdot E_1)$. Applying Corollary 2.3 to the log pair $(S_1, \lambda_1 C_d^1 + (\lambda_1 m_0 - 1)E_1)$ at the point $P_1$, we get

$$k\lambda_1 m_0 \geqslant k+1,$$

which gives $\lambda_1 \geqslant \frac{2k+1}{kd}$. Then $k \geqslant d-2$ by Lemma 3.1(ii). Since

$$k \leqslant C_d^1 \cdot E_1 = m_0 = d-1,$$

either $k = d-1$ or $k = d-2$. If $k = d-1$, then $C_d$ has singularity $\mathbb{K}_{d-1}$ at $P$. If $k = d-2$, then $C_d$ has singularity $\widetilde{\mathbb{K}}_{d-1}$ at the point $P$.

To complete the proof, we may assume that $L$ is an irreducible component of the curve $C_d$. Then $C_d = L + C_{d-1}$, where $C_{d-1}$ is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that $L$ is not its irreducible component. Denote by $C_{d-1}^1$ its proper transform on $S_1$. Put $n_0 = \mathrm{mult}_P(C_{d-1})$ and $n_1 = \mathrm{mult}_{P_1}(C_{d-1}^1)$. Then $n_0 = m_0 - 1 = d-2$ and $n_1 = m_1 - 1$. This implies that $P_1 \in C_{d-1}^1$, since the log pair $(S_1, \lambda_1 L^1 + (\lambda_1 m_0 - 1)E_1)$ is Kawamata log terminal at $P$. Hence, $n_1 \geqslant 1$. One the other hand, we have

$$d-1-n_0 = C_{d-1}^1 \cdot L^1 \geqslant n_1,$$

which implies that $n_0 + n_1 \leqslant d-1$. Then $n_1 = 1$, since $n_0 = d-2$.

We have $P_1 \in C_{d-1}^1$ and $C_{d-1}^1$ is smooth at $P_1$. Moreover, since

$$1 = d-1-n_0 = L^1 \cdot C_{d-1}^1 \geqslant n_1 = 1,$$

the curve $C_{d-1}^1$ intersects the curve $L^1$ transversally at the point $P_1$. Put $k = \mathrm{mult}_{P_1}(C_{d-1}^1 \cdot E_1)$. Then $k \geqslant 1$. Applying Lemma 2.2 to the log pair $(S_1, \lambda_1 C_{d-1}^1 + \lambda_1 L^1 + (\lambda_1(n_0+1)-1)E_1)$ at the point $P_1$, we get

$$k\Big(\lambda_1(n_0+2)-1\Big) + \lambda_1 \geqslant k+1.$$

Then $\lambda_1 \geqslant \frac{2k+1}{kd+1}$. Then $k \geqslant d-3$ by Lemma 3.1(iii). Since

$$k \leqslant E_1 \cdot C_{d-1}^1 = n_0 = d-2,$$

either $k = d-2$ or $k = d-3$. In the former case, $C_d$ has singularity $\mathbb{T}_{d-1}$ at the point $P$. In the latter case, $C_d$ has singularity $\widetilde{\mathbb{T}}_{d-1}$ at the point $P$. □

**Lemma 3.6.** Suppose that (**A**) holds and $m_0 \leqslant d-2$. Then the line $L$ is not an irreducible component of the curve $C_d$.

*Proof.* Suppose that $L$ is an irreducible component of the curve $C_d$. Let us see for a contradiction. Put $C_d = L + C_{d-1}$, where $C_{d-1}$ is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that $L$ is not its irreducible component. Denote by $C_{d-1}^1$ its proper transform on $S_1$. Put $n_0 = \mathrm{mult}_P(C_{d-1})$ and $n_1 = \mathrm{mult}_{P_1}(C_{d-1}^1)$. Then $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1 + \lambda_1 C_{d-1}^1)$ is not Kawamata log terminal at $P_1$ and is Kawamata log terminal outside of the point $P_1$. In particular, $n_1 \neq 0$, because $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1)$ is Kawamata log terminal at $P_1$. On the other hand,

$$d-1-n_0 = L^1 \cdot C_{d-1}^1 \geqslant n_1,$$

which implies that $n_0 + n_1 \leqslant d-1$. Furthermore, we have $n_0 = m_0 - 1 \leqslant d-3$.

Since $n_0 + n_1 \geqslant 2n_1$, we have $n_1 \leqslant \frac{d-1}{2}$. Then $\lambda n_1 < 1$ by Lemma 3.1(i). Thus, we can apply Theorem 2.16 to the log pair $(S_1, (\lambda_1(n_0+1)-1)E_1 + \lambda_1 L^1 + \lambda_1 C_{d-1}^1)$ at the point $P_1$. This gives either

$$\lambda_1\big(d-1-n_0\big) = \lambda_1 C_{d-1}^1 \cdot L^1 \geqslant 2\Big(2 - \lambda_1\big(n_0+1\big)\Big)$$

or

$$\lambda_1 n_0 = \lambda_1 C_{d-1}^1 \cdot E_1 \geqslant 2\big(1-\lambda_1\big)$$

(or both). In the former case, we have $\lambda_1(d+1+n_0) \geqslant 4$. In the latter case, we have $\lambda_1(n_0+2) > 2$. Thus, in both cases we have $\lambda_1(d-1) \geqslant 2$, since $n_0 \leqslant d-3$. But $\lambda_1(d-1) < 2$ by Lemma 3.1(i). This is a contradiction. □

If the curve $C_d$ is GIT-semistable, then $m_0 \leqslant d-2$ by Lemma 3.3. Thus, it follows from Lemma 3.5 that we may assume that

$$m_0 \leqslant d-2$$

in order to complete the proof of Theorems 1.10 and 1.15. Moreover, if $L$ is not an irreducible component of the curve $C_d$, then

$$d-m_0 = C_d^1 \cdot L^1 \geqslant m_1.$$

Thus, if (**A**) holds, then $m_0+m_1 \leqslant d$ by Lemma 3.6. Similarly, if the curve $C_d$ is GIT-semistable, then $m_0+m_1 \leqslant d$ by Lemma 3.3. Thus, to complete the proof of Theorems 1.10 and 1.15, we may also assume that

$$m_0+m_1 \leqslant d. \tag{3.7}$$

Then $\lambda(m_0+m_1)<3$ by Lemma 3.1(v), so that $(S_2, \lambda C_d^2+(\lambda m_0-1)E_1^2+(\lambda(m_0+m_1)-2)E_2)$ is Kawamata log terminal outside of the point $P_2$ by Lemma 2.13. Furthermore, we have

**Lemma 3.8.** Suppose that $P_2=E_1^2\cap E_2$. Then (**A**) does not hold and $C_d$ is GIT-unstable.

*Proof.* We have $m_0-m_1=E_1^2\cdot C_d^2\geqslant m_2$, so that

$$m_2 \leqslant \frac{m_0}{2}, \tag{3.9}$$

because $2m_2\leqslant m_1+m_2$. On the other hand, $m_0\leqslant d-2$ by assumption. Thus, we have $m_2\leqslant\frac{d-2}{2}$.

Suppose that (**A**) holds. Then $\lambda=\lambda_1$ and $\lambda_1 m_2<1$ by Lemma 3.1(v). Thus, we can apply Theorem 2.16 to the log pair $(S_2, \lambda_1 C_d^2+(\lambda_1 m_0-1)E_1^2+(\lambda_1(m_0+m_1)-2)E_2)$. This gives either

$$\lambda_1\big(m_0-m_1\big)=\lambda_1 C_d^2\cdot E_1^2\geqslant 2\Big(3-\lambda_1\big(m_0+m_1\big)\Big)$$

or

$$\lambda_1 m_1=\lambda_1 C_d^2\cdot E_2\geqslant 2\big(2-\lambda_1 m_0\big)$$

(or both). The former inequality implies $\lambda_1(3m_0+m_1)\geqslant 6$. The latter inequality implies $\lambda_1(2m_0+m_1)\geqslant 4$. On the other hand, $m_0+m_1\leqslant d$ by (3.7), and $m_0\leqslant d-2$ by assumption. Thus, $3m_0+m_1\leqslant 3d-4$ and $2m_0+m_1\leqslant 2d-2$. Then $\lambda_1(3m_0+m_1)<6$ by Lemma 3.1(vi), and $\lambda_1(2m_0+m_1)<4$ by Lemma 3.1(i). The obtained contradiction shows that (**A**) does not hold.

We see that (**B**) holds. We have to show that $C_d$ is GIT-unstable. Suppose that this is not the case, so that $C_d$ is GIT-semistable. Let us seek for a contradiction.

By Lemma 3.2, we have $2m_0+m_1+m_2\leqslant\frac{5d}{3}$, because

$$\mathrm{wt}_{(3,2)}\Big(f_d\big(x_1,x_2\big)\Big)=2m_0+m_1+m_2.$$

Thus, we have $\lambda_2(2m_0+m_1+m_2)-4<1$ by Lemma 3.1(v). Hence, the log pair $(S_3, \lambda_2 C_d^3+(\lambda_2 m_0-1)E_1^3+(\lambda_2(m_0+m_1)-2)E_2^3+(\lambda_2(2m_0+m_1+m_2)-4)E_3)$ is Kawamata log terminal outside of the point $P_3$ by Remark 2.10.

If $P_3=E_1^3\cap E_3$, then it follows from Theorem 2.12 that

$$\lambda_2\big(m_0-m_1-m_2\big)=\lambda_2 C_d^3\cdot E_1^3>5-\lambda_2\big(2m_0+m_1+m_2\big),$$

which implies that $m_0>\frac{5}{3\lambda_2}=\frac{2d}{3}$, which is impossible by Lemma 3.3. If $P_3=E_2^3\cap E_3$, then it follows from Theorem 2.12 that

$$\lambda_2\big(m_1-m_2\big)=\lambda_2 C_d^3\cdot E_2^3>5-\lambda_2\big(2m_0+m_1+m_2\big),$$

which implies that $m_0+m_1>\frac{5}{2\lambda_2}=d$, which is impossible by Lemma 3.3. Thus, we see that $P_3\notin E_1^3\cup E_2^3$. Then the log pair $(S_3, \lambda_2 C_d^3+(\lambda_2(2m_0+m_1+m_2)-4)E_3)$ is not Kawamata log terminal at $P_3$. Hence, Theorem 2.12 gives

$$\lambda_2 m_2=\lambda_2 C_d^3\cdot E_3>1,$$

which implies that $m_2 > \frac{1}{\lambda_2} = \frac{2d}{5}$. Then $m_0 > \frac{4d}{5}$ by (3.9), which is impossible by Lemma 3.3. $\square$

Thus, to complete the proof of Theorems 1.10 and 1.15, we may assume that

$$P_2 \neq E_1^2 \cap E_2.$$

Denote by $L^2$ the proper transform of the line $L$ on the surface $S_2$.

**Lemma 3.10.** One has $P_2 \neq L^2 \cap E_2$.

*Proof.* Suppose that $P_2 = L^2 \cap E_2$. If $L$ is not an irreducible component of the curve $C_d$, then

$$d - m_0 - m_1 = L^2 \cdot E_2 \geqslant m_2,$$

which implies that $m_0+m_1+m_2 \leqslant d$. Thus, if (**A**) holds, then $\lambda = \lambda_1$ and $L$ is not an irreducible component of the curve $C_d$ by Lemma 3.6, which implies that

$$\lambda_1 d \geqslant \lambda_1\big(m_0 + m_1 + m_2\big) > 3$$

by Lemma 2.14. On the other hand, $\lambda_1 d < 3$ by Lemma 3.1(iv). This shows that (**B**) holds.

Since $\lambda = \lambda_2 = \frac{5}{2d} < \frac{3}{d}$ and $\lambda_2(m_0+m_1+m_2) > 3$ by Lemma 2.14, we have $m_0+m_1+m_2 > d$. In particular, the line $L$ must be an irreducible component of the curve $C_d$.

Put $C_d = L + C_{d-1}$, where $C_{d-1}$ is a reduced curve in $\mathbb{P}^2$ of degree $d-1$ such that $L$ is not its irreducible component. Denote by $C_{d-1}^1$ its proper transform on $S_1$, and denote by $C_{d-1}^2$ its proper transform on $S_2$. Put $n_0 = \mathrm{mult}_P(C_{d-1})$, $n_1 = \mathrm{mult}_{P_1}(C_{d-1}^1)$ and $n_2 = \mathrm{mult}_{P_2}(C_{d-1}^2)$. Then $(S_2, (\lambda_2(n_0+n_1+2)-2)E_2 + \lambda_2 L^1 + \lambda_2 C_{d-1}^1)$ is not Kawamata log terminal at $P_2$ and is Kawamata log terminal outside of the point $P_2$. Then Theorem 2.12 implies

$$\lambda_2\big(d-1-n_0-n_1\big) = \lambda_2 C_{d-1}^2 \cdot L^2 > 1 - \big(\lambda_2(n_0+n_1+2)-2\big) = 3 - \lambda_2(n_0+n_1+2),$$

which implies that $\frac{5(d+1)}{2d} = \lambda_2(d+1) > 3$. Hence, $d=4$. Then $\lambda = \lambda_2 = \frac{5}{8}$.

By (3.7), $n_0 + n_1 \leqslant 2$. Thus, $n_0 = n_1 = n_2 = 1$, since

$$\frac{5}{8}\big(n_0+n_1+n_2+3\big) = \lambda_2\big(m_0+m_1+m_2\big) > 3$$

by Lemma 2.14. Then $C_3$ is a irreducible cubic curve that is smooth at $P$, the line $L$ is tangent to the curve $C_3$ at the point $P$, and $P$ is an inflexion point of the cubic curve $C_3$. This implies that $\mathrm{lct}_P(\mathbb{P}^2, C_d) = \frac{2}{3}$. Since $\frac{2}{3} > \frac{5}{8} = \lambda_2$, the log pair $(\mathbb{P}^2, \lambda_2 C_d)$ must be Kawamata log terminal at the point $P$, which contradicts (**B**). $\square$

Recall that $m_0 + m_1 \leqslant d$ by (3.7). Then $m_1 \leqslant \frac{d}{2}$, since $2m_1 \leqslant m_0 + m_1$. Thus, we have

$$\lambda\big(m_0+m_1+m_2\big) \leqslant \lambda\big(m_0+2m_1\big) \leqslant \lambda\frac{3d}{2} \leqslant \lambda_2\frac{3d}{2} = \frac{15}{4} < 4. \tag{3.11}$$

Therefore, the log pair $(S_3, \lambda C_d^3 + (\lambda(m_0+m_1)-2)E_2^3 + (\lambda(m_0+m_1+m_2)-3)E_3)$ is Kawamata log terminal outside of the point $P_3$ by Lemma 2.13.

**Lemma 3.12.** One has $P_3 \neq E_2^3 \cap E_3$.

*Proof.* If $P_3 = E_2^3 \cap E_3$, then Theorem 2.12 gives

$$\lambda\big(m_1 - m_2\big) = \lambda C_d^3 \cdot E_2^3 > 1 - \Big(\lambda\big(m_0+m_1+m_2\big) - 3\Big) = 4 - \lambda\big(m_0+m_1+m_2\big),$$

which implies that $\lambda(m_0+2m_1) > 4$. But $\lambda(m_0+2m_1) < 4$ by (3.11). $\square$

Let $f_4\colon S_4 \to S_3$ be a blow up of the point $P_3$, and let $E_4$ be its exceptional curve. Denote by $C_d^4$ the proper transform on $S_4$ of the curve $C_d$, denote by $E_3^4$ the proper transform on $S_4$ of the curve $E_3$, and denote by $L^4$ the proper transform of the line $L$ on the surface $S_4$. Then $(S_4, \lambda C_d^4 + (\lambda(m_0+m_1+m_2)-3)E_3^4 + (\lambda(m_0+m_1+m_2+m_3)-4)E_4)$ is not Kawamata log terminal at some point $P_4 \in E_4$ by Remark 2.10. Moreover, we have

$$2L^4+E_1+2E_2+E_3 \sim (f_1\circ f_2\circ f_3\circ f_4)^*\Big(\mathcal{O}_{\mathbb{P}^2}(2)\Big) - (f_2\circ f_3\circ f_4)^*\big(E_1\big) - (f_3\circ f_4)^*\big(E_2\big) - f_4^*\big(E_3\big) - E_4.$$

**Lemma 3.13.** The linear system $|2L^4+E_1+2E_2+E_3|$ is a pencil that does not have base points. Moreover, every divisor in $|2L^4+E_1+2E_2+E_3|$ that is different from $2L^4+E_1+2E_2+E_3$ is a smooth curve whose image on $\mathbb{P}^2$ is a smooth conic that is tangent to $L$ at the point $P$.

*Proof.* All assertions follows from $P_2 \notin E_1^2 \cup L^2$ and $P_3 \notin E_2^3$. $\square$

Let $C_2^4$ be a general curve in $|2L^4+E_1+2E_2+E_3|$. Denote by $C_2$ its image on $\mathbb{P}^2$, and denote by $\mathcal{L}$ the pencil generated by $2L$ and $C_2$. Then $P$ is the only base point of the pencil $\mathcal{L}$, and every conic in $\mathcal{L}$ except $2L$ and $C_2$ intersects $C_2$ at $P$ with multiplicity 4 (cf. [3, Remark 1.14]).

**Lemma 3.14.** One has $m_0+m_1+m_2+m_3 \leqslant m_0+m_1+2m_2 \leqslant \frac{5}{\lambda}$. If $m_0+m_1+m_2+m_3=\frac{5}{\lambda}$, then $d$ is even and $C_d$ is a union of $\frac{d}{2} \geqslant 2$ smooth conics in $\mathcal{L}$, where $d=4$ if (**A**) holds.

*Proof.* By (3.7), we have $m_2+m_3 \leqslant 2m_2 \leqslant m_0+m_1 \leqslant d$. This gives

$$m_0+m_1+m_2+m_3 \leqslant m_0+m_1+2m_2 \leqslant 2d=\frac{5}{\lambda_2} \leqslant \frac{5}{\lambda}.$$

To complete the proof, we may assume that $m_0+m_1+m_2+m_3=\frac{5}{\lambda}$. Then all inequalities above must be equalities. Thus, we have $m_2=m_3=\frac{d}{2}$ and $\lambda_1=\lambda_2$. In particular, if (**A**) holds, then $d=4$, because $\lambda_1<\lambda_2=\frac{5}{2d}$ for $d \geqslant 5$ by Lemma 3.1(vii). Moreover, since $m_0 \geqslant m_1 \geqslant m_2=\frac{d}{2}$ and $m_0+m_1 \leqslant d$, we see that $m_0=m_1=\frac{d}{2}$. Thus, $d$ is even and

$$C_d^4 \sim \frac{d}{2}\Big(2L^4+E_1+2E_2+E_3\Big),$$

where $d=4$ if (**A**) holds. Since $|2L^4+E_1+2E_2+E_3|$ is a free pencil and $C_d^4$ is reduced, it follows from Lemma 3.13 that $C_d^4$ is a union of $\frac{d}{2}$ smooth curves in $|2L^4+E_1+2E_2+E_3|$. In particular, $L^4$ is not an irreducible component of $C_d^4$. Thus, the curve $C_d$ is a union of $\frac{d}{2}$ smooth conics in $\mathcal{L}$, where $d=4$ if (**A**) holds. $\square$

We see that $m_0+m_1+m_2+m_3 \leqslant \frac{5}{\lambda}$. Moreover, if $m_0+m_1+m_2+m_3=\frac{5}{\lambda}$, then $C_d$ is an even Płoski curve. Furthermore, if $m_0+m_1+m_2+m_3=\frac{5}{\lambda}$ and (**A**) holds, then $d=4$. Thus, to prove Theorems 1.10 and 1.15, we may assume that

$$m_0+m_1+m_2+m_3<\frac{5}{\lambda}.$$

Let us show that this assumption leads to a contradiction. By Lemma 2.13, this inequality implies that the log pair $(S_4, \lambda C_d^4+(\lambda(m_0+m_1+m_2)-3)E_3^4+(\lambda(m_0+m_1+m_2+m_3)-4)E_4)$ is Kawamata log terminal outside of the point $P_4$.

**Lemma 3.15.** One has $P_4 \neq E_3^4 \cap E_4$.

*Proof.* By Lemma 3.14, $m_0+m_1+2m_2 \leqslant \frac{5}{\lambda}$. If $P_4=E_3^4 \cap E_4$, then Theorem 2.12 gives

$$\lambda\big(m_2-m_3\big)=\lambda C_d^4 \cdot E_3^4>5-\lambda\big(m_0+m_1+m_2+m_3\big),$$

which implies that $m_0+m_1+2m_2>\frac{5}{\lambda}$. This shows that $P_4 \neq E_3^4 \cap E_4$. $\square$

Thus, the log pair $(S_4, \lambda C_d^4+(\lambda(m_0+m_1+m_2+m_3)-4)E_4)$ is not Kawamata log terminal at $P_4$ and is Kawamata log terminal outside of the point $P_4$.

Let $Z^4$ be the curve in $|2L^4+E_1+2E_2+E_3|$ that passes through the point $P_4$. Then $Z^4$ is a smooth irreducible curve by Lemma 3.10. Denote by $Z$ the proper transform of this curve on $\mathbb{P}^2$. Then $Z$ is a smooth conic in the pencil $\mathcal{L}$ by Lemma 3.13. If $Z$ is not an irreducible component of the curve $C_d$, then

$$2d-\big(m_0+m_1+m_2+m_3\big)=Z^4 \cdot C_d^4 \geqslant \mathrm{mult}_{P_4}(C_d^4).$$

On the other hand, it follows from Lemma 2.14 that

$$\mathrm{mult}_{P_4}(C_d^4)+m_0+m_1+m_2+m_3>\frac{5}{\lambda}.$$

This shows that $Z$ is an irreducible component of the curve $C_d$, since $\lambda \leqslant \lambda_2 = \frac{5}{2d}$.

Put $C_d = Z + C_{d-2}$, where $C_{d-2}$ is a reduced curve in $\mathbb{P}^2$ of degree $d-2$ such that $Z$ is not its irreducible component. Denote by $C^1_{d-2}$, $C^2_{d-2}$, $C^3_{d-2}$ and $C^4_{d-2}$ its proper transforms on the surfaces $S_1$, $S_2$, $S_3$ and $S_4$, respectively. Put $n_0 = \mathrm{mult}_P(C_{d-2})$, $n_1 = \mathrm{mult}_{P_1}(C^1_{d-2})$, $n_2 = \mathrm{mult}_{P_2}(C^2_{d-2})$, $n_3 = \mathrm{mult}_{P_3}(C^3_{d-2})$ and $n_4 = \mathrm{mult}_{P_4}(C^4_{d-2})$. Then

$$\Big(S_4, \lambda C^4_{d-2} + \lambda Z^4 + (\lambda(n_0+n_1+n_2+n_3+4)-4)E_4\Big)$$

is not Kawamata log terminal at $P_4$ and is Kawamata log terminal outside of the point $P_4$. Thus, applying Theorem 2.12, we get

$$\lambda\Big(2\big(d-2\big)-n_0-n_1-n_2-n_3\Big) = \lambda C^4_{d-2}\cdot Z^4 > 5-\lambda\big(n_0+n_1+n_2+n_3+4\big),$$

which implies that $\lambda > \frac{5}{2d}$. This is impossible, since $\lambda \leqslant \lambda_2 = \frac{5}{2d}$.

The obtained contradiction completes the proof of Theorems 1.10 and 1.15.

## 4. Smooth surfaces in $\mathbb{P}^3$

The purpose of this section is to prove Theorem 1.17. Let $S$ be a smooth surface in $\mathbb{P}^3$ of degree $d \geqslant 3$, let $H_S$ be its hyperplane section, let $P$ be a point in $S$, let $T_P$ be the hyperplane section of the surface $S$ that is singular at $P$. Note that $T_P$ is reduced by Lemma 2.6. Put $\lambda = \frac{2d-3}{d(d-2)}$. Then Theorem 1.17 follows from Theorem 1.10, Remark 2.4 and

**Proposition 4.1.** Let $D$ be any effective $\mathbb{Q}$-divisor on $S$ such that $D \sim_{\mathbb{Q}} H_S$. Suppose that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve $T_P$. Then $(S, \lambda D)$ is log canonical at $P$.

For $d = 3$, this result is just [3, Corollary 1.13]. In the remaining part of the section, we will prove Proposition 4.1. Note that we will do this *without* using [3, Corollary 1.13]. Let us start with

**Lemma 4.2.** The following assertions hold:

- (i) $\lambda \leqslant \frac{2}{d-1}$,
- (ii) if $d \geqslant 5$, then $\lambda \leqslant \frac{3}{d+1}$,
- (iii) if $d \geqslant 5$, then $\lambda \leqslant \frac{4}{d+3}$,
- (iv) If $d \geqslant 6$, then $\lambda \leqslant \frac{3}{d+2}$,
- (v) $\lambda \leqslant \frac{4}{d+1}$,
- (vi) $\lambda \leqslant \frac{3}{d}$.

*Proof.* The equality $\frac{2}{d-1} = \lambda + \frac{d-3}{d(d-1)(d-2)}$ implies (i), $\frac{4}{d+1} = \lambda + \frac{d^2-5d+3}{d(d+1)(d-2)}$ implies (ii), and $\frac{4}{d+3} = \lambda + \frac{2d^2-11d+9}{d(d+3)(d-2)}$ implies (iii). Similarly, (iv) follows from $\frac{3}{d+2} = \lambda + \frac{d^2-7d+6}{d(d^2-4)}$, (v) follows from $\frac{4}{d+1} = \lambda + \frac{2d^2-7d+3}{d(d+1)(d-2)}$, and (vi) follows from $\frac{3}{d} = \lambda + \frac{d-3}{d(d-2)}$. $\square$

Let $n$ be the number of irreducible components of the curve $T_P$. Write

$$T_P = T_1 + \cdots + T_n,$$

where each $T_i$ is an irreducible curve on the surface $S$. For every curve $T_i$, we denote its degree by $d_i$, and we put $t_i = \mathrm{mult}_P(T_i)$.

**Lemma 4.3.** Suppose that $n \geqslant 2$. Then

$$T_i \cdot T_i = -d_i(d-d_i-1)$$

for every $T_i$, and $T_i \cdot T_j = d_i d_j$ for every $T_i$ and $T_j$ such that $T_i \neq T_j$.

*Proof.* The curve $T_P$ is cut out on $S$ by a hyperplane $H \subset \mathbb{P}^3$. Then $H \cong \mathbb{P}^2$. Hence, for every $T_i$ and $T_j$ such that $T_i \neq T_j$, we have $(T_i \cdot T_j)_S = (T_i \cdot T_j)_H = d_i d_j$. In particular, we have

$$d_1 = T_P \cdot T_1 = T_1^2 + \sum_{i=2}^{n} T_i \cdot T_1 = T_1^2 + \sum_{i=2}^{n} d_i d_1 = T_1^2 + (d - d_1)d_1,$$

which gives $T_1 \cdot T_1 = -d_1(d - d_1 - 1)$. Similarly, we see that $T_i \cdot T_i = -d_i(d - d_i - 1)$ for every curve $T_i$. □

Let $D$ be any effective $\mathbb{Q}$-divisor on $S$ such that $D \sim_{\mathbb{Q}} H_S$. Write

$$D = \sum_{i=1}^{n} a_i T_i + \Delta,$$

where each $a_i$ is a non-negative rational number, and $\Delta$ is an effective $\mathbb{Q}$-divisor on $S$ whose support does not contain the curves $T_1, \ldots, T_n$. To prove Proposition 4.1, it is enough to show that the log pair $(S, \lambda D)$ is log canonical at $P$ provided that at least one number among $a_1, \ldots, a_n$ vanishes.

Without loss of generality, we may assume that $a_n = 0$. Suppose that the log pair $(S, \lambda D)$ is not log canonical at $P$. Let us seek for a contradiction.

**Lemma 4.4.** Suppose that $n \geqslant 2$. Then

$$\sum_{i=1}^{k} a_i d_i d_n \leqslant d_n - t_n \mathrm{mult}_P(\Delta).$$

In particular, $\sum_{i=1}^{k} a_i d_i \leqslant 1$ and each $a_i$ does not exceed $\frac{1}{d_i}$.

*Proof.* One has

$$d_n = T_n \cdot D = T_n \cdot \left(\sum_{i=1}^{n} a_i T_i + \Delta\right) = \sum_{i=1}^{n} a_i d_i d_n + T_n \cdot \Delta \geqslant \sum_{i=1}^{n} a_i d_i d_n + t_n \mathrm{mult}_P(\Delta),$$

which implies the required inequality. □

Put $m_0 = \mathrm{mult}_P(D)$.

**Lemma 4.5.** Suppose that $P \in T_n$. Then $d_n > \frac{d-1}{2}$. If $n \geqslant 2$, then $T_n$ is smooth at $P$.

*Proof.* Since $T_n$ is not contained in the support of the divisor $D$, we have

$$d \geqslant d_n = T_n \cdot D \geqslant t_n m_0,$$

which implies that $m_0 \leqslant \frac{d_n}{t_n}$. Since $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, we have $d_n > \frac{d-1}{2}$ by Lemma 4.2(i). Moreover, if $n \geqslant 2$ and $t_n \geqslant 2$, then it follows from Lemma 2.5 that

$$\frac{1}{\lambda} < m_0 \leqslant \frac{d_n}{t_n} \leqslant \frac{d-1}{t_n} \leqslant \frac{d-1}{2},$$

which is impossible by Lemma 4.2(i). □

Now we are going to use Theorem 2.15 to prove

**Lemma 4.6.** Suppose that $n \geqslant 3$ and $P$ is contained in at least two irreducible components of the curve $T_P$ that are different from $T_n$ and that are both smooth at $P$. Then they are tangent to each other at $P$.

*Proof.* Without loss of generality, we may assume that $P \in T_1 \cap T_2$ and $t_1 = t_2 = 1$. Suppose that $T_1$ and $T_2$ are not tangent to each other at $P$. Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$, so that $D = a_1 T_1 + a_2 T_2 + \Omega$. Then $a_1 d_1 + a_2 d_2 \leqslant 1$ by Lemma 4.4.

Put $k_0 = \mathrm{mult}(\Omega)$. Then

$$d_1 + a_1 d_1 \big(d - d_1 - 1\big) - a_2 d_1 d_2 = \Omega \cdot T_1 \geqslant k_0$$

by Lemma 4.3. Similarly, we have

$$d_2 - a_1 d_1 d_2 + a_2 d_2\big(d - d_2 - 1\big) = \Omega \cdot T_2 \geqslant k_0.$$

Adding these two inequalities together and using $a_1 d_1 + a_2 d_2 \leqslant 1$, we get

$$2k_0 \leqslant d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) \leqslant d_1 + d_2 + \big(d - d_1 - d_2 - 1\big) = d - 1.$$

Thus, $k_0 \leqslant \frac{1}{\lambda}$ by Lemma 4.2(i).

Since $\lambda k_0 \leqslant 1$, we can apply Theorem 2.15 to the log pair $(S, \lambda a_1 T_1 + \lambda a_2 T_2 + \lambda\Omega)$ at the point $P$. This gives either $\lambda\Omega \cdot T_1 > 2(1 - \lambda a_2)$ or $\lambda\Omega \cdot T_2 > 2(1 - \lambda a_1)$. Without loss of generality, we may assume that $\lambda\Omega \cdot T_2 > 2(1 - \lambda a_1)$. Then

$$d_2 + a_2 d_2\big(d - d_2 - 1\big) - a_1 d_1 d_2 = \Omega \cdot T_2 > \frac{2}{\lambda} - 2a_1. \tag{4.7}$$

Applying Theorem 2.12 to the log pair $(S, \lambda a_1 T_1 + \lambda b_1 T_2 + \lambda\Omega)$ and the curve $T_1$ at the point $P$, we get

$$d_1 + a_1 d_1\big(d - d_1 - 1\big) = \Big(\lambda a_2 T_2 + \lambda\Omega\Big) \cdot T_1 > \frac{1}{\lambda}.$$

Adding this inequality to (4.7), we get

$$d + 1 \geqslant d - 1 + 2a_1 \geqslant d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) + 2a_1 > \frac{3}{\lambda},$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. Thus, it follows from Lemma 4.2(ii) that either $d = 3$ or $d = 4$.

If $d = 3$, then $n = 3$ and $d_1 = d_2 = d_3 = \lambda = 1$, which implies that $a_1 + a_2 > 1$ by (4.7). On the other hand, we know that $a_1 d_1 + a_2 d_2 \leqslant 1$, so that $a_1 + a_2 \leqslant 1$. This shows that $d \neq 3$.

We see that $d = 4$. Then $\lambda = \frac{5}{8}$ and $d_1 + d_2 \leqslant 3$. If $d_1 = d_1 = 1$, then (4.7) gives $2a_2 + a_1 > \frac{11}{5}$. If $d_1 = 1$ and $d_2 = 2$, then (4.7) gives $a_2 > \frac{3}{5}$. If $d_1 = 2$ and $d_2 = 1$, then (4.7) gives $a_2 > \frac{11}{5}$. All these three inequalities are inconsistent, because $a_1 d_1 + a_2 d_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. ☐

Note that every line contained in the surfaces $S$ that passes through $P$ must be an irreducible component of the curve $T_P$. Moreover, the curve $T_n$ cannot be a line by Lemma 4.5. Thus, Lemma 4.6 implies that there exists at most one line in $S$ that passes through $P$. In particular, we see that $n < d$.

**Lemma 4.8.** Suppose that $n \geqslant 3$ and $P$ is contained in at least two irreducible components of the curve $T_P$ that are different from $T_n$. Then these curves are smooth at $P$.

*Proof.* Without loss of generality, we may assume that $P \in T_1 \cap T_2$ and $t_1 \leqslant t_2$. We have to show that $t_1 = t_2 = 1$. We may assume that $d \geqslant 5$, because the required assertion is obvious in the cases $d = 3$ and $d = 4$.

Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$ and put $k_0 = \mathrm{mult}_P(\Omega)$. Then $m_0 = k_0 + a_1 t_1 + a_2 t_2$. Moreover, we have $a_1 d_1 + a_2 d_2 \leqslant 1$ by Lemma 4.4. On the other hand, it follows from Lemma 4.3 that

$$d - 1 \geqslant d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) = \Omega \cdot \Big(T_1 + T_2\Big) \geqslant k_0\big(t_1 + t_2\big),$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. Thus, we have $k_0 \leqslant \frac{d-1}{t_1+t_2}$. Hence, if $t_1 + t_2 \geqslant 4$, then

$$m_0 = k_0 + a_1 t_1 + a_2 t_2 \leqslant k_0 + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{t_1 + t_2} + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{t_1 + t_2} + 1 \leqslant \frac{d+3}{4}$$

because $a_1 d_1 + a_2 d_2 \leqslant 1$. Since $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, the inequality $m_0 \leqslant \frac{d+3}{4}$ gives $\lambda > \frac{d+3}{4}$, which is impossible by Lemma 4.2(iii). Thus, $t_1 + t_2 \leqslant 3$. Since $t_1 \leqslant t_2$, we have $t_1 = 1$ and $t_2 \leqslant 2$.

To complete the proof of the lemma, we have to prove that $t_2 = 1$. Suppose $t_2 \neq 1$. Then $t_2 = 2$, since $t_1 + t_2 \leqslant 3$. Since $k_0 \leqslant \frac{d-1}{t_1+t_2} = \frac{d-1}{3}$ and $a_1 d_1 + a_2 d_2 \leqslant 1$, we have

$$m_0 = k_0 + a_1 t_1 + a_2 t_2 \leqslant k_0 + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{32} + a_1 d_1 + a_2 d_2 \leqslant \frac{d-1}{t_1 + t_2} + 1 = \frac{d+2}{3}.$$

On the other hand, $m_0 > \frac{1}{\lambda}$ by Lemma 2.5, so that $\lambda > \frac{3}{d+2}$. Then $d = 5$ by Lemma 4.2(iv).

Since $d = 5$, $t_1 = 1$ and $t_2 = 2$, we have $n = 3$, $d_1 = 1$, $d_2 = 3$ and $d_3 = 1$. Applying Theorem 2.12 to the log pair $(S, \lambda a_1 T_1 + \lambda a_2 T_2 + \lambda\Omega)$, we get

$$1 + 3a_1 = d_1 + a_2 d_1\big(d - d_1 - 1\big) = \Big(\lambda a_2 T_2 + \lambda\Omega\Big) \cdot T_1 > \frac{1}{\lambda} = \frac{15}{7},$$

which gives $a_1 > \frac{8}{21}$. On the other hand, $a_1 + 3a_2 \leqslant 1$, because $a_1 d_1 + a_2 d_2 \leqslant 1$. Since $m_0 > \frac{1}{\lambda} = \frac{15}{7}$ by Lemma 2.5, we see that

$$\frac{15}{7} - \frac{1}{9} = \frac{128}{63} > \frac{8 - 5a_1}{3} = \frac{3 - a_1 + \frac{7(1-a_1)}{3}}{2} = \frac{3 - a_1 + 7a_2}{2} = \frac{3 - 3a_1 + 3a_2}{2} + a_1 + 2a_2 =$$

$$= \frac{\Delta \cdot T_2}{2} + a_1 + 2a_2 \geqslant \frac{\mathrm{mult}_P\Big(\Delta \cdot T_2\Big)}{2} + a_1 + 2a_2 \geqslant \frac{t_2 k_0}{2} + a_1 + 2a_2 = k_0 + a_1 + 2a_2 = m_0 > \frac{15}{7},$$

which is absurd. $\square$

Now we are ready to prove

**Lemma 4.9.** One has $m_0 \leqslant \frac{d+1}{2}$.

*Proof.* Suppose that $m_0 > \frac{d+1}{2}$. Let us seek for a contradiction. If $n = 1$, then

$$d = T_n \cdot D \geqslant 2m_0,$$

which implies that $m_0 \leqslant \frac{d}{2}$. Thus, have $n \geqslant 2$. Then $a_1 \leqslant \frac{1}{d_1}$ by Lemma 4.4. Moreover, either $t_n = 0$ or $t_n = 1$ by Lemma 4.5. Hence, there is an irreducible component of $T_P$ that passes through $P$ and is different from $T_n$, because $T_P$ is singular at $P$. Without loss of generality, we may assume that $t_1 \geqslant 1$.

Put $\Upsilon = \sum_{i=2}^{n} a_i T_i + \Delta$, so that $D = a_1 T_1 + \Upsilon$. Put $n_0 = \mathrm{mult}_P(\Upsilon)$, so that $m_0 = n_0 + a_1 t_1$. Then $t_n n_0 \leqslant d_n - a_1 d_1 d_n$ by Lemma 4.4, and

$$d_1 + a_1 d_1(d - d_1 - 1) = \Upsilon \cdot T_1 \geqslant t_1 n_0 \tag{4.10}$$

by Lemma 4.3. Adding these two inequalities, we get $(t_1 + t_n)n_0 \leqslant d_1 + d_n + a_1 d_1(d - d_1 - d_n - 1)$. Hence, if $n \geqslant 3$ and $t_n = 1$, then

$$2n_0 \leqslant \big(t_1 + t_n\big)n_0 \leqslant d_1 + d_n + a_1 d_1\big(d - d_1 - d_n - 1\big) \leqslant d - 1 \leqslant d - a_1 d_1,$$

because $a_1 \leqslant \frac{1}{d_1}$. Similarly, if $n = 2$ and $t_n = 1$, then

$$2n_0 \leqslant \big(t_1 + t_n\big)n_0 \leqslant d_1 + d_2 + a_1 d_1\big(d - d_1 - d_2 - 1\big) = d_1 + d_2 - a_1 d_1 = d - a_1 d_1.$$

Thus, if $t_n = 1$, then $n_0 \leqslant \frac{d - a_1 d_1}{2}$, which is impossible. Indeed, the inequality $n_0 \leqslant \frac{d - a_1 d_1}{2}$ gives

$$\frac{d+1}{2} < m_0 = n_0 + a_1 t_1 \leqslant n_0 + a_1 d_1 \leqslant \frac{d - a_1 d_1}{2} + a_1 d_1 = \frac{d + a_1 d_1}{2} \leqslant \frac{d+1}{2},$$

because $a_1 \leqslant \frac{1}{d_1}$. This shows that $t_n = 0$.

If $t_1 \geqslant 2$, then it follows from (4.10) that

$$\frac{d+1}{2} < m_0 \leqslant n_0 + a_1 d_1 \leqslant \frac{d_1 + a_1 d_1(d - d_1 - 1)}{2} + a_1 d_1 = \frac{d_1 + a_1 d_1(d - d_1 + 1)}{2} \leqslant \frac{d+1}{2},$$

because $a_1 \leqslant \frac{1}{d_1}$. This shows that $t_1 = 1$.

Since $t_1 = 1$ and $t_n = 0$, there exists an irreducible component of the curve $T_P$ that passes through $P$ and is different from $T_1$ and $T_n$. In particular, we have $n \geqslant 3$. Without loss of generality, we may assume $P \in T_2$. Then $T_2$ is smooth at $P$ by Lemma 4.8.

Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$ and put $k_0 = \mathrm{mult}_P(\Omega)$. Then $a_1 d_1 + a_2 d_2 \leqslant 1$ by Lemma 4.4. Thus, it follows from Lemma 4.3 that

$$2k_0 \leqslant \Omega \cdot \Big(T_1 + T_2\Big) = d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) \leqslant d - 1,$$

which implies $k_0 \leqslant \frac{d-1}{2}$. Then

$$\frac{d+1}{2} < m_0 = k_0 + a_1t_1 + a_2t_2 \leqslant k_0 + a_1d_1 + a_2d_2 \leqslant \frac{d-1}{2} + a_1d_1 + a_2d_2 \leqslant \frac{d-1}{2} + 1 = \frac{d+1}{2},$$

because $a_1d_1 + a_2d_2 \leqslant 1$. The obtained contradiction completes the proof of the lemma. $\square$

Let $f_1\colon S_1 \to S$ be a blow up of the point $P$, and let $E_1$ be its exceptional curve. Denote by $D^1$ the proper transform of the $\mathbb{Q}$-divisor $D$ on the surface $S_1$. Then

$$K_{S_1} + \lambda D^1 + \big(\lambda m_0 - 1\big)E_1 \sim_{\mathbb{Q}} f_1^*\Big(K_S + \lambda D\Big),$$

which implies that $(S_1, \lambda D^1 + (\lambda m_0 - 1)E_1)$ is not log canonical at some point $P_1 \in E_1$.

By Lemma 4.9, we have $m_0 \leqslant \frac{d+1}{2}$. By Lemma 4.2(v), we have $\lambda \leqslant \frac{4}{d+1}$. This gives $\lambda m_0 \leqslant 2$. Thus, the log pair $(S_1, \lambda D^1 + (\lambda m_0 - 1)E_1)$ is log canonical at every point of the curve $E_1$ that is different from $P_1$ by Lemma 2.13.

Put $m_1 = \mathrm{mult}_{P_1}(D^1)$. Then Lemma 2.5 gives

$$m_0 + m_1 > \frac{2}{\lambda}. \tag{4.11}$$

For each curve $T_i$, denote by $T_i^1$ its proper transform on $S_1$. Put $T_P^1 = \sum_{i=1}^n T_i^1$.

**Lemma 4.12.** One has $P_1 \notin T_P^1$.

*Proof.* Suppose that $P_1 \in T_P^1$. Let us seek for a contradiction. If $T_P$ is irreducible, then

$$d - 2m_0 = T_P^1 \cdot D^1 \geqslant m_1,$$

so that $m_1 + 2m_0 \leqslant d$. This inequality gives

$$\frac{3}{\lambda} < m_1 + 2m_0 \leqslant d,$$

because $2m_0 \geqslant m_0 + m_1 > \frac{2}{\lambda}$ by (4.11). This shows that $T_P$ is reducible, because $\lambda \leqslant \frac{3}{d}$ by Lemma 4.2(vi).

We see that $n \geqslant 2$. If $P_1 \in T_n^1$, then

$$d - 1 - m_0 \geqslant d_n - m_0 = d_n - m_0t_n = T_n^1 \cdot D^1 \geqslant m_1,$$

which is impossible, because $m_0 + m_1 > \frac{2}{\lambda}$ by (4.11), and $\lambda \leqslant \frac{2}{d-1}$ by Lemma 4.2(i). Thus, we see that $P_1 \notin T_n^1$.

Without loss of generality, we may assume that $P_1 \in T_1^1$. Put $\Upsilon = \sum_{i=2}^n a_iT_i + \Delta$, and denote by $\Upsilon^1$ the proper transform of the $\mathbb{Q}$-divisor $\Omega$ on the surface $S_1$. Put $n_0 = \mathrm{mult}_P(\Upsilon)$, put $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$ and put $t_1^1 = \mathrm{mult}_{P_1}(T_1^1)$. Then

$$d_1 + a_1d_1\big(d - d_1 - 1\big) - n_0t_1 = T_1^1 \cdot \Upsilon^1 \geqslant t_1^1 n_1,$$

which implies that $n_0t_1 + n_1t_1^1 \leqslant d_1 + a_1d_1(d - d_1 - 1)$.

Note that $t_1^1 \leqslant t_1$. Moreover, we have $a_1 \leqslant \frac{1}{d_1}$ by Lemma 4.4. Thus, if $t_1^1 \geqslant 2$, then

$$2\big(n_0 + n_1\big) \leqslant t_1^1\big(n_0 + n_1\big) \leqslant n_0t_1 + n_1t_1^1 \leqslant d_1 + a_1d_1\big(d - d_1 - 1\big) \leqslant d_1 + \big(d - d_1 - 1\big) = d - 1,$$

which implies that $n_0 + n_1 \leqslant \frac{d-1}{2}$. Moreover, if $n_0 + n_1 \leqslant \frac{d-1}{2}$, then it follows from (4.11) that

$$\frac{d+3}{2} = 2 + \frac{d-1}{2} \geqslant 2a_1d_1 + \frac{d-1}{2} \geqslant 2a_1t_1 + \frac{d-1}{2} \geqslant a_1\big(t_1 + t_1^1\big) + n_0 + n_1 = m_0 + m_1 > \frac{2}{\lambda}$$

which gives $d \leqslant 4$ by Lemma 4.2(iii). Thus, if $d \geqslant 5$, then $t_1^1 = 1$. Furthermore, if $d \leqslant 4$, then $d_1 \leqslant 3$, which implies that $t_1^1 \leqslant 1$. This shows that $t_1^1 = 1$ in all cases. Thus, the curve $T_1^1$ is smooth at $P_1$.

Applying Theorem 2.11 to the log pair $(S_1, \lambda\Upsilon^1 + \lambda a_1T_1^1 + (\lambda(n_0 + a_1t_1) - 1)E_1)$, we see that

$$\lambda\big(d - 1 - n_0t_1\big) \geqslant \lambda\Big(d_1 + a_1d_1\big(d - d_1 - 1\big) - n_0t_1\Big) = \lambda\Omega^1 \cdot T_1^1 > 2 - \lambda\big(n_0 + a_1t_1\big),$$

because $a_1 \leqslant \frac{1}{d_1}$. Thus, we have $d-1+a_1t_1-n_0(t_1-1) > \frac{2}{\lambda}$. But $m_0 = a_1t_1+n_0 > \frac{1}{\lambda}$ by Lemma 2.5. Adding these inequalities together, we obtain

$$d-1+2a_1t_1-n_0(t_1-2) > \frac{3}{\lambda}. \tag{4.13}$$

If $t_1 \geqslant 2$, this gives

$$d+1 \geqslant d-1+2a_1d_1 \geqslant d-1+2a_1t_1 \geqslant d-1+2a_1t_1-n_0(t_1-2) > \frac{3}{\lambda}.$$

because $a_1 \leqslant \frac{1}{d_1}$. One the other hand, if $d \geqslant 5$, then $\lambda \leqslant \frac{3}{d+1}$ by Lemma 4.2(ii). Thus, if $d \geqslant 5$, then $t_1 = 1$. Moreover, if $d = 3$, then $d_1 \leqslant 2$, which implies that $t_1 = 1$ as well. Furthermore, if $d = 4$ and $t_1 \neq 1$, then $d_1 = 3$, $t_1 = 2$, $\lambda = \frac{5}{8}$, which implies that

$$\frac{1}{3} = \frac{1}{d_1} \geqslant a_1 > \frac{9}{20}$$

by (4.13). Thus, we see that $t_1 = 1$ in all cases. This simply means that the curve $T_1$ is smooth at the point $P$.

Since $a_1 \leqslant \frac{1}{d_1}$, we have

$$d-1-n_0 \geqslant d_1 + a_1d_1\big(d-d_1-1\big) - n_0 = \Omega^1 \cdot T_1^1 \geqslant n_1,$$

which implies that $n_1 \leqslant \frac{n_0+n_1}{2} \leqslant \frac{d-1}{2}$. Then $\lambda n_1 \leqslant 1$ by Lemma 4.2(i). Hence, we can apply Theorem 2.15 to the log pair $(S_1, \lambda\Upsilon^1 + \lambda a_1 T_1^1 + (\lambda(n_0+a_1t_1)-1)E_1)$ at the point $P_1$. This gives either

$$\Upsilon^1 \cdot T_1^1 > \frac{4}{\lambda} - 2(n_0+a_1)$$

or $\Upsilon^1 \cdot E_1 > \frac{2}{\lambda} - 2a_1$ (or both). Since $a_1 \leqslant \frac{1}{d_1}$, the former inequality gives

$$d-1-n_0 \geqslant d_1 + a_1d_1\big(d-d_1-1\big) - n_0 = \Upsilon^1 \cdot T_1^1 > \frac{4}{\lambda} - 2(n_0+a_1).$$

Similarly, the latter inequality gives

$$n_0 = \lambda\Upsilon^1 \cdot E_1 > \frac{2}{\lambda} - 2a_1.$$

Thus, either $d-1+2a_1+n_0 > \frac{4}{\lambda}$ or $2a_1+n_0 > \frac{2}{\lambda}$ (or both).

If $t_n \geqslant 1$, then $d_n \neq 1$ by Lemma 4.5. Thus, if $t_n \geqslant 1$, then

$$d-1 \geqslant d_n \geqslant a_1d_1d_n + n_0 \geqslant 2a_1 + n_0$$

by Lemma 4.4. Therefore, if $t_n \geqslant 1$, then

$$2(d-1) \geqslant d-1+2a+n_0 > \frac{4}{\lambda}$$

or $d-1 \geqslant 2a+n_0 > \frac{2}{\lambda}$, because $d-1+2a+n_0 > \frac{4}{\lambda}$ or $2a+n_0 > \frac{2}{\lambda}$. In both cases, we get $\lambda > \frac{d-1}{2}$, which is impossible by Lemma 4.2(i). This shows that $t_n = 0$, so that $P \notin T_n$.

Since $T_1$ is smooth at $P$ and $P \notin T_n$, there must be another irreducible component of $T_P$ passing through $P$ that is different from $T_1$ and $T_n$. In particular, we see that $n \geqslant 3$. Without loss of generality, we may assume that $P \in T_2$. Then $T_2$ is smooth at $P$ by Lemma 4.8, so that $t_2 = 1$. Moreover, the curves $T_1$ and $T_2$ are tangent at $P$ by Lemma 4.6, which implies that $d \geqslant 4$. Since $P_1 \in T_1^1$, we see that $P_1 \in T_2^1$ as well.

Put $\Omega = \sum_{i=3}^n a_iT_i + \Delta$ and $k_0 = \mathrm{mult}_P(\Omega)$, so that $m_0 = k_0 + a_1 + a_2$. Then $a_1d_1 + a_2d_2 \leqslant 1$ by Lemma 4.4.

Denote by $\Omega^1$ the proper transform of the $\mathbb{Q}$-divisor $\Omega$ on the surface $S_1$. Put $k_1 = \mathrm{mult}_{P_1}(\Omega^1)$. Then

$$d-1-2k_0 \geqslant d_1+d_2+\big(a_1d_1+a_2d_2\big)\big(d-d_1-d_2-1\big) - 2k_0 = \Omega^1 \cdot \Big(T_1^1+T_2^1\Big) \geqslant 2k_1$$

because $a_1d_1+a_2d_2\leqslant 1$ and $d\geqslant d_1+d_2+d_n\geqslant d_1+d_2+1$. This gives $k_0+k_1\leqslant\frac{d-1}{2}$. On the other hand, we have

$$2a_1+2a_2+k_0+k_1=m_0+m_1>\frac{2}{\lambda}$$

by (4.11). Thus, we have

$$\frac{d+3}{2}=2+\frac{d-1}{2}\geqslant 2\big(a_1d_1+a_2d_2\big)+\frac{d-1}{2}\geqslant 2a_1+2a_2+\frac{d-1}{2}\geqslant 2a_1+2a_2+k_0+k_1>\frac{2}{\lambda}$$

because $a_1d_1+a_2d_2\leqslant 1$. By Lemma 4.2(iii) this gives $d=4$. Thus, we have $\lambda=\frac{5}{8}$.

Since $d=4>n\geqslant 3$, we have $n=3$. Without loss of generality, we may assume that $d_1\leqslant d_2$. By Lemma 4.6, there exists at most one line in $S$ that passes through $P$. This shows that $d_1=1$, $d_2=2$ and $d_3=1$. Thus, $T_1$ and $T_3$ are lines, $T_2$ is a conic, $T_1$ is tangent to $T_2$ at $P$, and $T_3$ does not pass through $P$. In particular, the curves $T_1^1$ and $T_1^2$ intersect each other transversally at $P_1$.

By Lemma 4.3, we have $T_1\cdot T_1=T_2\cdot T_2=-2$ and $T_1\cdot T_2=2$. On the other hand, the log pair $(S_1,\lambda a_1T_1^1+\lambda a_2T_2^1+\lambda\Omega^1+(\lambda(a_1+a_2+k_0)-1)E_1)$ is not log canonical at the point $P_1$. Thus, applying Theorem 2.11 to this log pair and the curve $T_1^1$, we get

$$\lambda\big(1+2a_1-2a_2-k_0\big)=\lambda\Omega^1\cdot T_1^1>2-\lambda(a_1+a_2+k_0)-\lambda a_2,$$

which implies that $3a_1>\frac{2}{\lambda}-1=\frac{11}{5}$, because $\lambda=\frac{5}{8}$. Similarly, applying Theorem 2.11 to this log pair and the curve $T_2^1$, we get

$$\lambda\big(2-2a_1+2a_2-k_0\big)=\lambda\Omega^1\cdot T_2^1>2-\lambda(a_1+a_2+k_0)-\lambda a_1,$$

which implies that $3a_2>\frac{2}{\lambda}-2=\frac{6}{5}$. Hence, we have $a_1>\frac{11}{15}$ and $a_2>\frac{2}{5}$, which is impossible, since $a_1+2a_2=a_1d_1+a_2d_2\leqslant 1$. The obtained contradiction completes the proof of the lemma. ☐

Now we are going to show that the curve $T_P$ has at most two irreducible components. This follows from

**Lemma 4.14.** One has $n\geqslant 2$ and $\mathrm{mult}_P(T_P)=2$. Moreover, if $n=2$, then $P\in T_1\cap T_2$, both curves $T_1$ and $T_2$ are smooth at $P$, and $d_1\leqslant d_2$.

*Proof.* If $T_P$ is irreducible and $\mathrm{mult}_P(T_P)\geqslant 3$, then Lemma 2.5 gives

$$d=T_P\cdot D\geqslant 3m_0>\frac{3}{\lambda},$$

which is impossible by Lemma 4.2(vi). Thus, if $n=1$, then $\mathrm{mult}_P(T_P)=2$.

To complete the proof, we may assume that $n\geqslant 2$. Then $t_n=0$ or $t_n=1$ by Lemma 4.5. In particular, there exists an irreducible component of the curve $T_P$ different from $T_n$ that passes through $P$. Without loss of generality, we may assume that $P\in T_1$.

Put $\Upsilon=\sum_{i=2}^{n}a_iT_i+\Delta$, and denote by $\Upsilon^1$ the proper transform of the $\mathbb{Q}$-divisor $\Omega$ on the surface $S_1$. Put $n_0=\mathrm{mult}_P(\Upsilon)$. Then the log pair $(S_1,\lambda\Upsilon^1+(\lambda(n_0+a_1t_1)-1)E_1)$ is not log canonical at $P_1$, since $P_1\not\in T_1^1$ by Lemma 4.12. In particular, it follows from Theorem 2.12 that

$$\lambda n_0=\lambda\Upsilon^1\cdot E_1>1,$$

which implies that $n_0>\frac{1}{\lambda}$. Thus, if $t_1\geqslant 2$, then it follows from Lemma 4.3 that

$$\frac{1}{\lambda}\geqslant\frac{d-1}{2}\geqslant\frac{d_1+a_1d_1(d-d_1-1)}{2}=\frac{\Upsilon\cdot T_1}{2}\geqslant\frac{t_1n_0}{2}\geqslant n_0>\frac{1}{\lambda},$$

because $a_1\leqslant\frac{1}{d_1}$ by Lemma 4.4, and $\lambda\leqslant\frac{2}{d-1}$ by Lemma 4.2(i). This shows that $t_1=1$, so that the curve $T_1$ is smooth at $P$.

If $t_n=1$ and $n\geqslant 3$, then

$$\frac{2}{\lambda}\geqslant d-1\geqslant d_1+d_n+ad_1(d-d_1-d_n-1)=\Upsilon\cdot\Big(T_1+T_n\Big)\geqslant 2n_0>\frac{2}{\lambda}.$$

Thus, if $t_n = 1$, then $n = 2$. Vice versa, if $n = 2$, then $t_n = 1$, because $T_1$ is smooth at $P$. Furthermore, if $n = 2$, then $d_1 \leqslant d_n$, because $d_n > \frac{d-1}{2}$ by Lemma 4.5. Therefore, to complete the proof, we must show that $n = 2$.

Suppose that $n \geqslant 3$. Let us seek for a contradiction. We know that $P \notin T_n$, so that $t_n = 0$. Then every irreducible component of the curve $T_P$ that contain $P$ is smooth at $P$ by Lemma 4.8. Hence, there should be at least one irreducible component of the curve $T_P$ containing $P$ that is different from $T_1$ and $T_n$. Without loss of generality, we may assume that $P \in T_2$.

Put $\Omega = \sum_{i=3}^{n} a_i T_i + \Delta$ and $k_0 = \mathrm{mult}_P(\Omega)$. By Lemma 4.4, we have $a_1 d_1 + a_2 d_2 \leqslant 1$. Thus, it follows from Lemma 4.3 that

$$2k_0 \leqslant \Delta \cdot \Big(T_1 + T_2\Big) = d_1 + d_2 + \big(a_1 d_1 + a_2 d_2\big)\big(d - d_1 - d_2 - 1\big) \leqslant d_1 + d_2 + \big(d - d_1 - d_2 - 1\big) = d - 1.$$

Hence, we have $k_0 \leqslant \frac{d-1}{2}$.

Denote by $\Omega^1$ the proper transform of the $\mathbb{Q}$-divisor $\Omega$ on the surface $S_1$. Then the log pair $(S_1, \lambda\Omega^1 + (\lambda(k_0 + a_1 + a_2) - 1)E_1)$ is not log canonical at $P_1$, because $P_1 \notin T_1^1$ and $P_1 \notin T_2^1$ by Lemma 4.12. In particular, it follows from Theorem 2.11 that

$$\lambda k_0 = \lambda \Omega^1 \cdot E_1 > 1,$$

which implies that $k_0 > \frac{1}{\lambda}$. This contradicts Lemma 4.2(i), because $k_0 \leqslant \frac{d-1}{2}$. ☐

Later, we will need the following simple

**Lemma 4.15.** Suppose that $d = 4$. Then $m_0 \leqslant \frac{11}{5}$.

*Proof.* If $n = 1$, then

$$2t_n \geqslant d_n = T_n \cdot D \geqslant t_n m_0,$$

so that $m_0 \leqslant 2 < \frac{11}{5}$. Thus, we may assume that $n \neq 1$. Then it follows from Lemma 4.14 that $n = 2$, $P \in T_1 \cap T_2$, both curves $T_1$ and $T_2$ are smooth at $P$, and $d_1 \leqslant d_2$.

If $d_2 = 2$, then $m_0 \leqslant 2 < \frac{11}{5}$, because

$$2 = T_2 \cdot D \geqslant m_0.$$

Thus, we may assume that $d_2 \neq 2$. Then $d_1 = 1$ and $d_2 = 3$. Then $\mathrm{mult}_P(\Delta) + 3a_1 \leqslant 3$ by Lemma 4.4. Moreover, we have

$$1 + 2a_1 = T_1 \cdot \Delta \geqslant \mathrm{mult}_P(\Delta).$$

The obtained inequalities give $m_0 = \mathrm{mult}_P(\Delta) + a_1 \leqslant \frac{11}{5}$. ☐

Let $f_2\colon S_2 \to S_1$ be a blow up of the point $P_1$. Denote by $E_2$ the $f_2$-exceptional curve, denote by $E_1^2$ the proper transform of the curve $E_1$ on the surface $S_2$, and denote by $D^2$ the proper transform of the $\mathbb{Q}$-divisor $D$ on the surface $S_2$. Then

$$K_{S_2} + \lambda D^2 + \big(\lambda m_0 - 1\big)E_1^2 + \Big(\lambda\big(m_0 + m_1\big) - 2\Big)E_2 \sim_{\mathbb{Q}} f_2^*\Big(K_{S_1} + \lambda D^1 + \big(\lambda m_0 - 1\big)E_1\Big).$$

By Remark 2.10, the log pair $(S_2, \lambda D^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at some point $P_2 \in E_1$.

**Lemma 4.16.** One has $m_0 + m_1 \leqslant \frac{3}{\lambda}$.

*Proof.* Suppose that $m_0 + m_1 > \frac{3}{\lambda}$. Then $2m_0 \geqslant m_0 + m_1 > \frac{3}{\lambda}$. But $m_0 \leqslant \frac{d+1}{2}$ by Lemma 4.9. Then $\lambda > \frac{3}{d+1}$. Thus, we have $d \leqslant 4$ by Lemma 4.2(ii). Moreover, if $d = 4$, then

$$\frac{22}{5} \geqslant 2m_0 \geqslant m_0 + m_1 > \frac{3}{\lambda} = \frac{24}{5}$$

by Lemma 4.15. This shows that $d = 3$.

We have $\lambda = 1$. If $n = 1$, then

$$3 = T_P \cdot D \geqslant 2m_0 \geqslant m_1 + m_0 > \frac{3}{\lambda} = 3,$$

which is absurd. Hence, it follows from Lemma 4.14 that $n = 2$, $d_1 = 1$, $d_2 = 2$ and $P \in T_1 \cap T_2$.

We have $m_0 = \mathrm{mult}_P(\Delta)+a_1$. On the other hand, we have $\mathrm{mult}_P(\Delta)+2a_1 \leqslant 2$ by Lemma 4.4. Moreover, we have

$$1 + a_1 = T_1 \cdot \Omega \geqslant \mathrm{mult}_P(\Delta),$$

which implies that $\mathrm{mult}_P(\Delta) - a_1 \leqslant 1$. Adding these inequalities, we get

$$3 \geqslant 2\mathrm{mult}_P(\Delta) + a = \mathrm{mult}_P(\Delta) + m_0 \geqslant m_1 + m_0 > \frac{3}{\lambda} = 3,$$

because $\mathrm{mult}_P(\Delta) \geqslant m_1$, since $P_1 \notin T_1^1$ by Lemma 4.12. $\square$

Thus, the log pair $(S_2, \lambda D^2 + (\lambda m_0 - 1)E_1^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is log canonical at every point of the curve $E_2$ that is different from the point $P$ by Lemma 2.13.

**Lemma 4.17.** One has $P_2 \neq E_1^2 \cap E_2$.

*Proof.* Suppose that $P_2 = E_1^2 \cap E_2$. Then Theorem 2.11 gives

$$\lambda\big(m_0 - m_1\big) = \lambda D^2 \cdot E_1^2 > 3 - \lambda\big(m_0 + m_1\big),$$

which implies that $m_0 > \frac{3}{2\lambda}$. But $m_0 \leqslant \frac{d+1}{2}$ by Lemma 4.9. Therefore, we have $\lambda > \frac{3}{d+1}$, which implies that $d \leqslant 4$ by Lemma 4.2(ii). If $d = 4$, then

$$\frac{12}{5} = \frac{3}{2\lambda} < m_0 \leqslant \frac{11}{5}$$

by Lemma 4.15. Thus, we have $d = 3$.

One has $\lambda = 1$. If $n = 1$, then

$$3 = T_P \cdot D \geqslant 2m_0 > \frac{3}{\lambda} = 3,$$

which is absurd. Hence, it follows from Lemma 4.14 that $n = 2$, $d_1 = 1$, $d_2 = 2$ and $P \in T_1 \cap T_2$.

We have $m_0 = \mathrm{mult}_P(\Delta) + a_1$. Moreover, we have $\mathrm{mult}_P(\Delta) + 2a_1 \leqslant 2$ by Lemma 4.4, Then $2\mathrm{mult}_P(\Delta) + a_1 \leqslant 3$, because

$$1 + a_1 = T_1 \cdot \Delta \geqslant \mathrm{mult}_P(\Delta).$$

Denote by $\Delta^1$ the proper transform of the divisor $\Delta$ on the surface $S_1$, and denote by $\Delta^2$ the proper transform of the divisor $\Delta$ on the surface $S_2$. Then $m_1 = \mathrm{mult}_{P_1}(\Delta^1)$, because $P_1 \notin T_1^1$ by Lemma 4.12. Thus, the log pair $(S_2, \lambda\Delta^2 + (m_0 - 1)E_1^2 + (m_0 + m_1 - 2)E_2)$ is not log canonical at $P_2$. Applying Theorem 2.11 to this pair and the curve $E_1^2$, we get

$$\mathrm{mult}_P(\Delta) - m_1 = \Delta^2 \cdot E_1^2 > 3 - m_0 - m_1,$$

which implies that $2\mathrm{mult}_P(\Delta) + a_1 > 3$. The latter is impossible, because we already proved that $2\mathrm{mult}_P(\Delta) + a_1 \leqslant 3$. $\square$

Thus, the log pair $(S_2, \lambda D^2 + (\lambda(m_0 + m_1) - 2)E_2)$ is not log canonical at $P_2$. Then Lemma 2.5 gives

$$m_0 + m_1 + m_2 > \frac{3}{\lambda}. \tag{4.18}$$

Denote by $T_P^2$ the proper transform of the curve $T_P$ on the surface $S^2$. Then

$$T_P^2 + E_1^2 \sim (f_1 \circ f_2)^*(\mathcal{O}_S(1)) - f_2^*(E_1) - E_2,$$

because $T_P^1 \sim f_1^*(\mathcal{O}_S(1)) - 2E_1$ by Lemma 4.14, and $P_1 \notin T_P^1$ by Lemma 4.12.

**Lemma 4.19.** The linear system $|T_P^2 + E_1^2|$ is a pencil that does not have base points in $E_2$.

*Proof.* Since $|T_P^1+E_1|$ is a two-dimensional linear system that does not have base points, $|T_P^2+E_1^2|$ is a pencil. Let $C$ be a curve in $|T_P^1+E_1|$ that passes through $P_1$ and is different from $T_P^1+E_1$. Then $C$ is smooth at $P$, since $P \in f_1(C)$ and $f_1(C)$ is a hyperplane section of the surface $S$ that is different from $T_P$. Since $C \cdot E_1 = 1$, we see that $T_P^1+E_1$ and $C$ intersect transversally at $P_1$. Thus, the proper transform of the curve $C$ on the surface $S_2$ is contained in $|T_P^1+E_1|$ and have no common points with $T_P^2+E_1^2$ in $E_2$. This shows that the pencil $|T_P^1+E_1|$ does not have base points in $E_2$. □

Let $Z^2$ be the curve in $|T_P^2+E_2|$ that passes through the point $P_2$. Then

$$Z^2 \neq T_P^2+E_1^2,$$

because $P_2 \neq E_1^2 \cap E_2$ by Lemma 4.17. Then $Z_2$ is smooth at $P_2$. Put $Z = f_1 \circ f_2(Z^2)$ and $Z^1 = f_2(Z^2)$. Then $P \in Z$ and $P_1 \in Z^1$. Moreover, the curve $Z$ is smooth at $P$, and the curve $Z_1$ is smooth at $P_1$. Furthermore, the curve $Z$ is reduced by Lemma 2.6.

The log pair $(S, \lambda Z)$ is log canonical at $P$, because $Z$ is smooth at $P$. Note that

$$Z \sim_{\mathbb{Q}} D.$$

Thus, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve $Z$ by Remark 2.4. Denote this irreducible component by $\overline{Z}$, and denote its degree in $\mathbb{P}^3$ by $\bar{d}$. Then $\bar{d} \leqslant d$.

**Lemma 4.20.** One has $P \notin \overline{Z}$.

*Proof.* Suppose that $P \in \overline{Z}$. Let us seek for a contradiction. Denote by $\overline{Z}^2$ the proper transform of the curve $\overline{Z}$ on the surface $S_2$. Then

$$d - m_0 - m_1 \geqslant \bar{d} - m_0 - m_1 = \overline{Z}^2 \cdot D^2 \geqslant m_2,$$

which implies that $m_0+m_1+m_2 \leqslant d$. One the other hand, $m_0+m_1+m_2 > \frac{3}{\lambda}$ by (4.18). This gives $\lambda > \frac{3}{d}$, which is impossible by Lemma 4.2(vi). □

In particular, the curve $Z$ is reducible. Denote by $\widehat{Z}$ its irreducible component that passes through $P$, denote its proper transform on the surface $S_1$ by $\widehat{Z}^1$, and denote its proper transform on the surface $S_2$ by $\widehat{Z}^2$. Then $\overline{Z} \neq \widehat{Z}$, $P_1 \in \widehat{Z}^1$ and $P_2 \in \widehat{Z}^2$. Denote by $\hat{d}$ the degree of the curve $\widehat{Z}$ in $\mathbb{P}^3$. Then $\hat{d}+\bar{d} \leqslant d$. Moreover, the intersection form of the curves $\widehat{Z}$ and $\overline{Z}$ on the surface $S$ is given by

**Lemma 4.21.** One has $\overline{Z} \cdot \overline{Z} = -\bar{d}(d-\bar{d}-1)$, $\widehat{Z} \cdot \widehat{Z} = -\hat{d}(d-\hat{d}-1)$ and $\overline{Z} \cdot \widehat{Z} = \bar{d}\hat{d}$.

*Proof.* See the proof of Lemma 4.3. □

Put $D = a\widehat{Z} + \Omega$, where $a$ is a positive rational number, and $\Omega$ is an effective $\mathbb{Q}$-divisor on the surface $S$ whose support does not contain the curve $\widehat{Z}$. Denote by $\Omega^1$ the proper transform of the divisor $\Omega$ on the surface $S_1$, and denote by $\Omega^2$ the proper transform of the divisor $\Omega$ on the surface $S_2$. Put $n_0 = \mathrm{mult}_P(\Omega)$, $n_1 = \mathrm{mult}_{P_1}(\Omega^1)$ and $n_2 = \mathrm{mult}_{P_2}(\Omega^2)$. Then $m_0 = n_0 + a$, $m_1 = n_1 + a$ and $m_2 = n_2 + a$. Then the log pair $(S_2, \lambda a\widehat{Z}^2 + \lambda\Omega^2 + (\lambda(n_0+n_1+2a)-2)E_2)$ is not log canonical at $P_2$, because $(S_2, \lambda D^2 + (\lambda(m_0+m_1)-2)E_2)$ is not log canonical at $P_2$. Thus, applying Theorem 2.11, we see that

$$\lambda\Big(\Omega \cdot \widehat{Z} - n_0 - n_1\Big) = \lambda\Omega^2 \cdot Z^2 > 1 - \Big(\lambda\big(n_0+n_1+2a\big) - 2\Big) = 3 - \lambda\big(n_0+n_1+2a\big),$$

which implies that

$$\Omega \cdot \widehat{Z} > \frac{3}{\lambda} - 2a. \qquad (4.22)$$

On the other hand, we have

$$\bar{d} = D \cdot \overline{Z} = \Big(a\widehat{Z} + \Omega\Big) \cdot \overline{Z} \geqslant a\widehat{Z} \cdot \overline{Z} = a\hat{d}\bar{d}$$

by Lemma 4.21. This gives

$$a \leqslant \frac{1}{\hat{d}}. \tag{4.23}$$

Thus, it follows from (4.22), (4.23) and Lemma 4.21 that

$$\frac{3}{\lambda} - 2 \leqslant \frac{3}{\lambda} - 2a < \Omega \cdot \widehat{Z} = \hat{d} + a\hat{d}\Big(d - \hat{d} - 1\Big) \leqslant d - 1,$$

which implies that $\lambda > \frac{3}{d+1}$. Then $d \leqslant 4$ by Lemma 4.2(ii).

**Lemma 4.24.** One has $d \neq 4$.

*Proof.* Suppose that $d = 4$. Then $\lambda = \frac{5}{8}$ and $\hat{d} \leqslant 3$. By Lemma 4.12, $\widehat{Z}$ is not a line, since every line passing through $P$ must be an irreducible component of the curve $T_P$. Thus, either $\widehat{Z}$ is a conic or $\widehat{Z}$ is a plane cubic curve. If $\widehat{Z}$ is a conic, then $\widehat{Z}^2 = -2$ and $a \leqslant \frac{1}{2}$ by (4.23). Thus, if $\widehat{Z}$ is a conic, then

$$2 + 2a = \Omega \cdot \widehat{Z} > \frac{3}{\lambda} - 2a = \frac{24}{5} - 2a,$$

which implies that $\frac{1}{2} \geqslant a > \frac{7}{10}$. This shows that $\widehat{Z}$ is a plane cubic curve. Then $\widehat{Z}^2 = 0$. Since $a \leqslant \frac{1}{3}$ by (4.23), we have

$$3 = \Omega \cdot \widehat{Z} > \frac{3}{\lambda} - 2a = \frac{24}{5} - 2a \geqslant \frac{24}{5} - \frac{2}{3} = \frac{62}{15},$$

which is absurd. □

Thus, we see that $d = 3$. Then $\widehat{Z}$ us either a line or a conic. But every line passing through $P$ must be an irreducible component of $T_P$. Since $\widehat{Z}$ is not an irreducible component of $T_P$ by Lemma 4.12, the curve $\widehat{Z}$ must be a conic. Then $\widehat{Z}^2 = 0$. Therefore, it follows from (4.22) that

$$3 - 2a = \frac{3}{\lambda} - 2a < \Omega \cdot \widehat{Z} = \hat{d} + a\hat{d}\Big(d - \hat{d} - 1\Big) = \hat{d} = 2,$$

which implies that $a > \frac{1}{2}$. But $a \leqslant \frac{1}{\hat{d}} = \frac{1}{2}$ by (4.23). The obtained contradiction completes the proof of Theorem 1.17.

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