INFINITELY RAMIFIED PROCESSES AND BRANCHING LÉVY PROCESSES

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An infinitely ramified process is a continuous-time process taking values in the space of point measures, such that every discrete-time skeleton is a branching random walk. We prove that such a process can be represented as a branching Lévy process : a particle process in which every individual produces offspring and moves according to some Poissonian dynamic. This result can be thought of as a branching process analogue of the correspondence between Lévy processes and infinitely divisible random variables. In the rest of the abstract, we give a more precise description of this result.

We denote by D the set of dyadic rational numbers, and by $\mathcal{P}$ the set of point measures that put finite mass to $[0, +\infty)$. This set is identified with the set of non-increasing sequences $(x_j) \in (\mathbb{R} \cup \{-\infty\})^{\mathbb{N}}$ that converge to $-\infty$ observing that

$$\mu \in \mathcal{P} \iff \mu = \sum_{j \in \mathbb{N}} \delta_{x_j}, \text{ with } x_1 \geq x_2 \geq \cdots \text{ and } \lim_{n \to +\infty} x_j = -\infty.$$

We introduce the shift operator $\tau_x \mu = \sum_{j \in \mathbb{N}} \delta_{x_j + x}$ on $\mathcal{P}$.

Definition 1 (Infinitely ramified process)**.** Let μ be a random variable taking values in $\mathcal{P}$. A branching random walk is a process $(Z_n, n \in \mathbb{N})$ that can be defined as follows: given $(\mu_{n,j}, j, n \in \mathbb{N})$ i.i.d. random variables with same law as μ, we set

$$Z_0 = \delta_0 \quad \text{and} \quad Z_{n+1} = \sum_{j=1}^{n} \tau_{z_{n,j}} \mu_{(n+1),j},$$

where $(z_{n,j}, j \in \mathbb{N})$ is the sequence of atoms associated to Z_n.

An infinitely ramified process is a process $(Z_t : t \in D)$, taking values in $\mathcal{P}$ such that for any $n \in \mathbb{N}$, $(Z_{k2^{-n}}, k \in \mathbb{N})$ is a branching random walk.

The precise definition of branching Lévy processes being a little bit technical, we only give here a rough picture. Let $\sigma^2 > 0$, $a \in \mathbb{R}$ and Λ a measure on $\mathcal{P}$ such that for some $\theta > 0$,

$$\int_{\mathcal{P}} \sum_{j \geq 1} e^{\theta x_j} - 1 - \theta x_1 \mathbf{1}_{|x_1| \leq 1} \Lambda(d(x_j)) < +\infty.$$

Let N be a Poisson point process with intensity $dt\Lambda$, we define N^{jump} and N^{repro} the image measures of N by the applications $(t, (x_j)) \mapsto (t, x_1)$ and $(t, (x_j)) \mapsto (t, (x_{j+1}))$ respectively.A branching Lévy process with characteristics (σ^2, a, Λ) is a particle process on $\mathbb{R}$ in which each individual moves according to a Brownian motion with variance σ^2, drift a, and jump measure N^{jump}. For each atom of $(t, (y_j))$ of N^{repro}, the individual produces offspring at time t at distance $y_1, y_2, \ldots$ from its current position.

Theorem 1. *Let Z be an infinitely ramified process. If there exists $\theta > 0$ such that $\mathbb{E}\left(\int e^{\theta x} dZ_1\right)$, then there exists a unique càdlàg extension of Z to a process on $\mathbb{R}_+$. Moreover, this càdlàg extension is a branching Lévy process with characteristics (σ^2, a, Λ).*