CONVEX LOG-SOBOLEV INEQUALITIES ON THE REAL LINE

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Let $H : \mathbb{R} \to [0, \infty)$ be a symmetric convex function, which is quadratic near zero. Suppose moreover that for some $A \in [1, \infty)$ and $\alpha \in (1, 2]$ we have

$$\forall_{x\in\mathbb{R}}\ \forall_{s\in[0,1]} \quad H(sx) \le As^{\alpha} H(x).$$

We say that a probability measure μ on $\mathbb{R}$ satisfies the modified *convex log-Sobolev inequality* (with constant $c < \infty$) if

$$\mathrm{Ent}_{\mu}(e^f) \le \int_{\mathbb{R}} H(cf')e^f d\mu \tag{1}$$

for every smooth *convex* Lipschitz function $f : \mathbb{R} \to \mathbb{R}$.

By considering only convex functions, we can work with measures which satisfy quite weak regularity conditions—most importantly, their supports do not need to be connected, which is not the case for many classical functional inequalities. On the other hand, a disturbing issue arises: the convex log-Sobolev inequality yields via standard reasonings only deviation inequalities for the upper tail of the functions, i.e.

$$\mu^{\otimes n}\big(\{x \in \mathbb{R}^n : f(x) \ge \int_{\mathbb{R}^n} f\ d\mu^{\otimes n} + t\}\big), \quad t \ge 0$$

(in the classical setting of smooth functions one obtains bounds on the lower tail simply by working with $-f$ instead of f, but in our situation $-f$ is usually not convex).

The goal of this talk is to overcome this problems by giving a sufficient and necessary condition for a probability measure μ on the real line to satisfy the convex log-Sobolev inequality. The condition is expressed in terms of the unique left-continuous and non-decreasing map transporting the symmetric exponential measure onto the reference measure μ. The proof will be based on the theory of weak-transport cost (see [2, 1]).

Based on joint work with Yan Shu (Université Paris Ouest Nanterre La Défense) [3].

References

[1] N. Gozlan, C. Roberto, P.M. Samson, Y. Shu, and P. Tetali, *Characterization of a class of weak transport-entropy inequalities on the line*, to appear in Ann. Inst. Henri Poincaré Probab. Stat., `https://arxiv.org/abs/1509.04202v2`.

[2] N. Gozlan, C. Roberto, P.M. Samson, and P. Tetali, *Kantorovich duality for general transport costs and applications*, to appear in J. Funct. Anal., `https://arxiv.org/abs/1412.7480v4`.

[3] Y. Shu, and M. Strzelecki *A characterization of a class of convex log-Sobolev inequalities on the real line*, preprint (2017), `https://arxiv.org/abs/1702.04698`.