LOCAL LAWS FOR SPECTRUM OF RANDOM MATRICES

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We present some recent results obtained jointly with F. Götze, A. Naumov and D. Timushev. We shall consider local laws for the classical ensembles of random matrices: Wigner matrices, sample covariance matrices and Girko – Ginibre matrices. We start from Wigner matrices. We consider a random symmetric matrix $\mathbf{X} = [X_{jk}]_{j,k=1}^{n}$ with upper triangular entries being i.i.d. random variables with mean zero and unit variance. We additionally suppose that $\mathbf{E}|X_{11}|^{4+\delta} =: \mu_{4+\delta} < \infty$ for some $\delta > 0$. We show that with high probability the typical distance between the Stieltjes transform of the empirical spectral distribution (ESD) of the matrix $n^{-\frac{1}{2}}\mathbf{X}$ and Wigner's semicircle law is of order $(nv)^{-1}\log n$, where v denotes the distance to the real line in the complex plane. We apply this result to estimate the rate of convergence of the ESD to the distribution function of the semicircle law as well as to establish the rigidity of the eigenvalues and the eigenvector delocalization. The result on the delocalization is optimal by comparison with GOE ensembles. Furthermore the techniques of this paper provide a new shorter proof for the optimal $O(n^{-1})$ rate of convergence of the expected ESD to the semicircle law.

We have proved the similar results for the sample covariance matrices. Consider a family $\mathcal{X} = \{X_{jk}\}$, $1 \le j \le n, 1 \le k \le p$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any $n \ge 1$ and $p \ge 1$. Introduce the matrices

$$\mathbf{X} = \frac{1}{\sqrt{p}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{pmatrix}$$

and corresponding sample covariance matrices

$$\mathbf{W} = \mathbf{X}\mathbf{X}^*.$$

Here and in what follows we denote by $\mathbf{A}^*$ the complex conjugate of matrix $\mathbf{A}$. The matrix $\mathbf{W}$ has a random spectrum $\{s_1^2, \dots, s_n^2\}$ and an associated spectral empirical distribution function $\mathcal{F}_n(x) = \frac{1}{n}\operatorname{card}\{j \le n : s_j^2 \le x\}, \quad x \in \mathbb{R}$. Averaging over the random values $X_{ij}(\omega)$, define the expected (non-random) empirical distribution functions $F_n(x) = \mathbf{E}\,\mathcal{F}_n(x)$. Let $G_y(x)$ denote the Marchenko – Pastur distribution function with parameter $y := n/p$ with density $g_y(x) = G_y'(x) = \frac{1}{2y\pi x}\sqrt{(x-a^2)(b^2-x)}\mathbb{I}_{[a^2,b^2]}(x)$, where $\mathbb{I}_{[a^2,b^2]}(x)$ denotes the indicator–function of the interval $[a^2, b^2]$ and $a^2 = (1-\sqrt{y})^2$ and $b^2 = (1+\sqrt{y})^2$. We show that with high probability the typical distance between the Stieltjes transform of the ESD of the matrix $\mathbf{W}$ and Marchenko – Pastur law is of order $(nv)^{-1}\log n$. We apply this result to estimate the rate of convergence of the ESD to the distribution function of the Pastur law as well as to establish the rigidity of the eigenvalues and the eigenvector delocalization. The result on the delocalization is optimal by comparison with LOE ensembles.

We consider as well the so-called Ginibre – Girko matrices and the product of independent matrices of Ginibre – Girko. Consider a family $\mathcal{X} = \{X_{jk}\}$, $1 \le j, k \le n$, of independent real random variables defined on some probability space $(\Omega, \mathfrak{M}, \Pr)$, for any

$n \geq 1$. Introduce the matrices

$$\mathbf{X} = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}.$$

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of matrix $\mathbf{X}$. We define the spectral measure μ_n on the complex plane $\mathcal{C}$ as

$$\mu_n(A) = \frac{1}{n}\sum_{j=1}^n \mathbb{I}_A(\lambda_j),$$

for any Borel set $A \subset \mathcal{C}$.

Theorem 1. *Assume that X_{jk}, for $j, k = 1, \ldots, n$ satisfy the following condition*

$$\max_{1\le j,k\le n} \mathbf{E}|X_{jk}|^8 \le C,$$

with some constant C independent on n. Then for any $d \in (0, \frac{1}{2}]$, any $\tau > 0$ small enough such that $|1 - |z|| \ge \tau$, and for any $D > 0$,

$$\mathbb{P}\Big\{\Big|\frac{1}{n}\sum_{j=1}^n f_{z_0}(\lambda_j) - \frac{1}{\pi}\int \mathbb{I}\{|z| \le 1\} f_{z_0}(z) dz d\overline{z}\Big| \le Cn^{-1+2d}\log^4 n \|\Delta f\|_{L_1}\Big\} \ge 1 - Cn^{-D},$$

with some constant C depending on D and τ. Function $f_{z_0}(z)$ is described in [3, Theorem 1.1].

We consider now independent $n \times n$ random matrices $\mathbf{X}^{(\alpha)} = [X_{jk}^{(\alpha)}]_{j,k=1}^n$, $\alpha = 1, \ldots, m$. We assume that $X_{jk}, 1 \le j, k \le n$, $\alpha = 1, \ldots, m$, are mutually independent random variables with $\mathbf{E}X_{jk}^{(\alpha)} = 0$ and $\mathbf{E}{X_{jk}^{(\alpha)}}^2 = 1$. Denote by $\lambda_1, ..., \lambda_n$ – the eigenvalues of the matrix $\mathbf{X} := \frac{1}{n^{\frac{m}{2}}}\prod_{\alpha=1}^m \mathbf{X}^{(\alpha)}$ and introduce the eigenvalue counting measure

$$\mu_n(A) := \frac{1}{n}\sum_{j=1}^n \mathbb{I}[\lambda_j \in A],$$

where A is a Borel set in $\mathbb{C}$. It is well known that the limit distribution of $\mu_n(\cdot)$ is the distribution of m-powered random variable uniform distributed in the unit circle. We shall interesting in the local behaviour of the distribution of eigenvalues of matrix $\mathbf{X}$. Our result is following

Theorem 2. *Assume that $X_{jk}^{(\alpha)}$, for $j, k = 1, \ldots, n$ and $\alpha = 1, \ldots, m$ satisfy the following condition*

$$\max_{1\le\alpha\le m}\max_{1\le j,k\le n} \mathbf{E}|X_{jk}^{(\alpha)}|^{16} \le C,$$

with some constant C independent on n. Then for any $d \in (0, \frac{1}{2}]$, any $\tau > 0$ small enough such that $|1 - |z|| \ge \tau$, and for any $D > 0$,

$$\mathbb{P}\Big\{\Big|\frac{1}{n}\sum_{j=1}^n f_{z_0}(\mu_j) - \frac{1}{m\pi}\int \mathbb{I}\{|z| \le 1\} f_{z_0}(z)|z|^{\frac{2}{m}-2} dz d\overline{z}\Big| \le Cn^{-1+2d}\log^4 n \|\Delta f\|_{L_1}\Big\} \ge 1 - Cn^{-D},$$

with some constant C depending on D and τ.

Our approach to the proof of the local limit theorems for the distribution of the spectrum of the product of matrices is different to the proof in the paper of Nemish. We reduce the problem to the rate of the convergence of the spectrum distribution of Hermitian block matrices of special type.

References

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