ON THE HCM PROPERTY FOR POWERS OF POSITIVE STABLE RANDOM VARIABLES, A COMPLEMENT TO BONDESSON'S CONJECTURE

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The class of generalized gamma convolutions (GGC) was introduced by O. Thorin and L. Bondesson [1] in order to describe weak limits of convolutions of gamma distributions. During the last decade, the interest for this class increased since the works of James, Roynette, Vallois and Yor for instance [3], [5]. All GGC distributions admit a density function, form a proper subclass of the positive self-decomposable distributions, and hence, are infinitely divisible. GGC distributed random variables could be also characterized through the hyperbolic complete monotonicity (HCM) property of their Laplace transforms widely studied in [1]. The HCM class of distributions corresponds to distributions with HCM density functions and forms a proper subclass of GGC.

All positive α-stable random variables $\mathbb{S}_\alpha$, $\alpha \in (0,1)$, are recognized to have a GGC distribution despite the fact that there is no closed form for the density of $\mathbb{S}_\alpha$, except for $\alpha = 1/2$. Lennart Bondesson conjectured in 1977 that a positive stable random variable $\mathbb{S}_\alpha$ has a hyperbolically completely monotone density (HCM) if and only if $\alpha \le 1/2$. In 2015, Bosch and Simon solved this problem and raised a more general conjecture: the density function of $\mathbb{S}_\alpha^p$ is an HCM function if and only if $\alpha \le 1/2$ and $|p| \ge \alpha/(1-\alpha)$. We give stronger credit to the second conjecture by investigating the explicitness of the value minimal power p_α for which the distribution of $\mathbb{S}_\alpha^{p_\alpha}$ is HCM and we also investigate whenever powers of positive stable random variables are generalized Gamma convolutions. In a joint work with Thomas Simon [4], we proved that for $p > 0$,

$$\text{the r.v. } \mathbb{S}_\alpha^{-p} \text{ is infinitely divisible if and only if } p \ge \alpha/\beta.$$

In our current work, we prove amongst others, the following stronger result:

Theorem 1. $\mathbb{S}_\alpha^p \sim \mathrm{HCM}$ *if and only if* $0 < \alpha \le 1/2$ *and* $|p|$ *is greater that some critical value* p_α *which belongs to* $[\frac{\alpha}{1-\alpha}, 2\alpha]$. *In particular, Bondesson's Conjecture is true.*

References

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