Dichotomy of global capacity density

Hiroaki Aikawa (Hokkaido University)

Probability and Analysis 2017

Banach Center in Będlewo
15–19 May 2017
Talk on 15 May 2017

Contents

Introduction

Let $\varphi \geq 0$ be an outer measure on $\mathbb{R}^n$, $n \geq 2$. Define

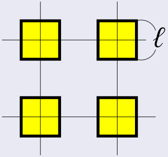
$$\mathcal{D}(\varphi, E, r) = \inf_{x \in \mathbb{R}^n} \frac{\varphi(E \cap B(x,r))}{\varphi(B(x,r))}, \quad \overline{\mathcal{D}}(\varphi, E, r) = \sup_{x \in \mathbb{R}^n} \frac{\varphi(E \cap B(x,r))}{\varphi(B(x,r))}.$$

- If E is bounded, then $\mathcal{D}(\varphi, E, r) = 0$.
- If $\mathcal{D}(\varphi, E, r) > 0$, then E is uniformly distributed at scale r wrt φ.

Proposition 1.1 (No dichotomy for Lebesgue measure)

Let $0 < \ell < 1$ *and let* $E = \bigcup_{x \in \mathbb{Z}^n} Q(x, \ell)$. *Then*

$$\lim_{r \to \infty} \mathcal{D}(m, E, r) = \lim_{r \to \infty} \overline{\mathcal{D}}(m, E, r) = \ell^n.$$



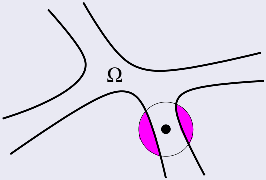
BMOA and Dichotomy of log-capacity density

Analytic f on $\mathbb{D}$ is BMOA $\overset{def}{\Longleftrightarrow} \sup_{|\zeta|<1} \|f_\zeta\|_2 < \infty$, $f_\zeta(z) = f\left(\dfrac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)$.

Theorem A (Hayman-Pommerenke [HP78])

Let $f : \mathbb{D} \to \Omega \subset \mathbb{C}$ be analytic. Then TFAE:

- *f is BMOA.*
- $^\exists R, {}^\exists \eta > 0$ *s.t.* $C_\ell(B(w,R) \setminus \Omega) > \eta$ *($^\forall w$).*
- $\mathcal{D}(C_\ell, E, R) > 0$.



Theorem B (Stegenga [Ste80])

Logarithmic capacity density enjoys the dichotomy:

$$\lim_{r\to\infty} \mathcal{D}(C_\ell, E, r) = \begin{cases} 0 & \text{if } \mathcal{D}(C_\ell, E, r) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \mathcal{D}(C_\ell, E, r)) > 0 \text{ for some } r > 0. \end{cases}$$

- See Bañuelos-Øksendal [BØ87] for harmonic morphisms, $n \geq 3$.

Another work of Hayman

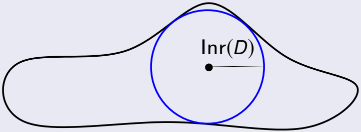
Definition 2.1 (Principal Frequency)

The first eigenvalue, i.e., $\inf \operatorname{spec}(-\Delta_D)$, *Rayleigh Ratio,*

$$\lambda(D) = \inf \left\{ \frac{\int_D |\nabla \varphi|^2 dx}{\|\varphi\|_2^2} : \varphi \in C_0^\infty(D) \right\}.$$

Definition 2.2 (Inradius)

$$\operatorname{Inr}(D) = \sup \left\{ r > 0 : {}^{\exists}B(x,r) \subset D \right\}.$$



Theorem C (Hayman [Hay78])

If $D \subset \mathbb{C}$ *simply connected, then* $\dfrac{A^{-1}}{\operatorname{Inr}(D)^2} \le \lambda(D) \le \dfrac{A}{\operatorname{Inr}(D)^2}$.

Higher dimensional case

$\operatorname{Inr}(D)$ is fragile; i.e., if $D' = D \setminus E$, polar, then $\lambda(D') = \lambda(D)$ and yet $\operatorname{Inr}(D')$ can be arbitrarily small. Some condition on ∂D is necessary.

- For $n \geq 3$ simple connectedness is irrelevant.

Theorem D (Hardy's inequality & CDC, Ancona [Anc86])

Define $\operatorname{Cap}_U(E) = \inf\{\|\nabla\varphi\|_2^2 : \varphi \geq 1 \text{ on } E, \varphi \in C_0^\infty(U)\}$.

Capacity Density Condition: $\dfrac{\operatorname{Cap}_{B(x,2r)}(B(x,r) \setminus D)}{\operatorname{Cap}_{B(x,2r)}(B(x,r))} \geq A \quad \text{for } {}^\forall x \in \partial D, {}^\forall r > 0$

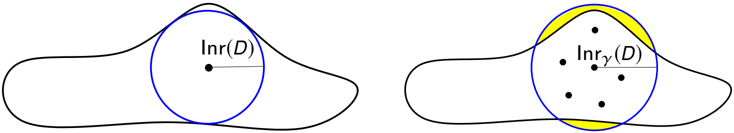
$\implies$ *Hardy's inequality:* $\displaystyle\int_D \frac{\varphi^2}{\delta_D^2} dx \leq \frac{1}{\varepsilon} \int_D |\nabla\varphi|^2 dx \quad \text{for } \varphi \in C_0^\infty(D).$

$\implies \dfrac{A^{-1}}{\operatorname{Inr}(D)^2} \leq \lambda(D) \leq \dfrac{A}{\operatorname{Inr}(D)^2}.$

- *If* $D \subset \mathbb{C}$ *is simply connected* $\implies$ *CDC* $\implies$ *Hardy's I.,* $\lambda(D) \approx \dfrac{1}{\operatorname{Inr}(D)^2}$.

Maz'ya-Shubin [MS05]: $n \geq 3$, γ-negligible sets ($0 < \gamma < 1$)

$$\mathrm{Inr}_\gamma(D) := \sup\left\{ r > 0 : \frac{C_2(B(x,r) \setminus D)}{C_2(B(x,r))} < \gamma \quad \text{for some } x \in D \right\}.$$



Theorem E (Maz'ya-Shubin [MS05])

There exists $A = A(\gamma, n) > 1$ such that

$$\frac{A^{-1}}{\mathrm{Inr}_\gamma(D)^2} \leq \lambda(D) \leq \frac{A}{\mathrm{Inr}_\gamma(D)^2} \quad \textit{for arbitrary } {}^\forall D.$$

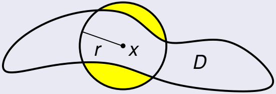
In particular, $\lambda(D) > 0 \iff \mathrm{Inr}_\gamma(D) < \infty$ *for some (all)* $\gamma \in (0,1)$.

- $\mathrm{Inr}_\gamma(D) < \infty \iff \lambda(D) > 0 \iff \mathrm{Inr}_{\gamma'}(D) < \infty$.
- *If* $\mathrm{Inr}_\gamma(D \setminus B(0,R)) \downarrow 0$, *then* $\mathrm{ess\,spec}(-\Delta_D) = \emptyset$ *(see [Per60]).*

Capacitary width and applications

Definition 3.1 (Capacitary width [Aik98])

Define capacitary width $w_\eta(D)$ with $0 < \eta < 1$ by



$$w_\eta(D) = \inf\left\{r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \setminus D)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} \ge \eta \quad \textit{for } {}^\forall x \in D\right\}.$$

Proposition 3.1 (Capacitary width and generalized inradius)

$$w_\eta(D) = \widetilde{\mathrm{Inr}}_\eta(D) := \sup\left\{r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \setminus D)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} < \eta \quad \textit{for } {}^\exists x \in D\right\}.$$

If $R > w_\eta(D)$, then $R > \widetilde{\mathrm{Inr}}_\eta(D)$, and vice versa.

Definition 3.2 (Torsion function & Survival probability)

Define Torsion function by

$$v_D(x) = \int_D G_D(x,y)dy = \int_0^\infty dt \int_D p_D(t,x,y)dy = \mathbb{E}^x[\tau_D]$$

with $\tau_D = \inf\{t \geq 0 : B_t \in \mathbb{R}^n \setminus D\}$, *Survival probability by*

$$P_D(t,x) = \int_D p_D(t,x,y)dy = \mathbb{P}^x(\tau_D > t).$$

- v_D solves the Saint-Venant problem, i.e., $-\Delta v_D = 1$ in D, $v_D = 0$ on ∂D.

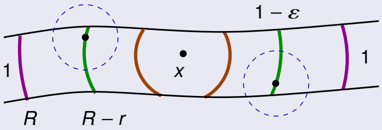
Theorem F (Bañuelos-van den Berg-Carroll [BnC94], [vdBC09], [vdB12])

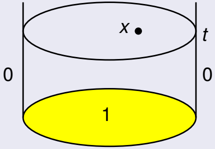
$$\frac{1}{\|v_D\|_\infty} \leq \lambda(D) \leq \frac{4 + 3n\log 2}{\|v_D\|_\infty}.$$

Torsion function: $v_D(x) = \int_D G_D(x,y)dy = \int_0^\infty dt \int_D p_D(t,x,y)dy = \mathbb{E}^x[\tau_D]$ with $\tau_D = \inf\{t \ge 0 : B_t \in \mathbb{R}^n \setminus D\}$.
Survival probability: $P_D(t,x) = \int_D p_D(t,x,y)dy = \mathbb{P}^x(\tau_D > t)$.

Theorem 3.2 (Elliptic and Parabolic estimates by capacitary width)

1. $A^{-1} w_\eta(D)^2 \le \|v_D\|_\infty \le A w_\eta(D)^2$.
2. $^\exists A_0, A_1 > 0$ *s.t.* $\omega^x(D \cap \partial B(x,R); D \cap B(x,R)) \le A_0 \exp\left(-\dfrac{A_1 R}{w_\eta(D)}\right)$.





3. $^\exists A_2, A_3 > 0$ *such that* $P_D(t,x,D) = \displaystyle\int_D p_D(t,x,y)dy$ *satisfies*

$$P_D(t,x,D) \le A_2 \exp\left(-\frac{A_3 t}{w_\eta(D)^2}\right) \quad \text{for } t > 0.$$

- Application of $\|v_D\|_\infty \approx w_\eta(D)^2$.

Theorem G (Bañuelos-van den Berg-Carroll [BnC94], [vdBC09], [vdB12])

$$\frac{1}{\|v_D\|_\infty} \le \lambda(D) \le \frac{4+3n\log 2}{\|v_D\|_\infty} \implies \lambda(D) \approx w_\eta(D)^{-2}.$$

Theorem H (Cranston-McConnell inequality [CM83])

If $D \subset \mathbb{C}$, *then* $\sup_{x\in D} \frac{1}{h(x)} \int_D G_D(x,y)h(y)dy \le A|D|$ *for* $^\forall h > 0$ *superharmonic.*

For $n \ge 3$ some regularity of D is necessary. Basic estimate

$$\sup_{x\in D} \frac{1}{h(x)} \int_D G_D(x,y)h(y)dy \le 4\sum_j \sup_{x\in D_j} \int_{D_j} G_{D_j}(x,y)dy = 4\sum_j \|v_{D_j}\|_\infty$$

with $D_j = \{x : 2^{j-1} < h(x) < 2^{j+2}\}$.

- Chung [Chu84], Bañuelos [Bañ92], A-Murata [AM96], [Aik98].
- Intrinsically Ultracontractivity (IU) $\implies$ Cranston-McConnell.

Intrinsic Ultracontractivity (IU)

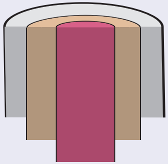
Davies-Simon [DS84] defined: D is Intrinsically Ultracontractive (IU) if

1. $-\Delta u = \lambda u$ in D with $u = 0$ on ∂D has the first eigenvalue $\lambda_D > 0$ and the eigenfunction $\varphi_D > 0$ normalized by $\|\varphi_D\|_2 = 1$.
2. If $t > 0$, then $c_t\varphi_D(x)\varphi_D(y) \le p_D(t,x,y) \le C_t\varphi_D(x)\varphi_D(y)$ for all $x, y \in D$, where c_t and C_t depend on t.

Theorem 3.3 (Unified condtions for GBHP and IU [Aik15])

Suppose $\lim_{R\to\infty} w_\eta(D \setminus \overline{B}(0,R)) = 0$. *Let* $g(x) = G(x, x_0)$ *and* $w_\eta(g < t) = w_\eta(\{x \in D : g(x) < t\})$.

1. $\displaystyle\int_0^1 w_\eta(g < t)^2 \frac{dt}{t} < \infty \implies$ *IU.*
2. $\displaystyle\int_0^1 w_\eta(g < t) \frac{dt}{t} < \infty \implies$ *GBHP.*



Parabolic Box.

Capacitary width and dichotomy of classical capacity

We have $\lambda(D) \approx \|v_D\|_\infty^{-2} \approx w_\eta(D)^{-2}$. How does $w_\eta(D)$ depend on $\eta \in (0,1)$?

Theorem 4.1 (Dependency on η, [Aik98, Proposition 2])

If $0 < \eta < \eta' < 1$, then $w_\eta(D) \le w_{\eta'}(D) \le A w_\eta(D)$. ($\Longrightarrow$ dichotomy)

Observe $\widetilde{\mathrm{Inr}}_\eta(D) \le \widetilde{\mathrm{Inr}}_{\eta'}(D) \le A\widetilde{\mathrm{Inr}}_\eta(D)$ since

$$w_\eta(D) = \inf\left\{r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \setminus D)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} \ge \eta \quad \text{for } {}^\forall x \in D\right\}$$

$$= \widetilde{\mathrm{Inr}}_\eta(D) = \sup\left\{r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \setminus D)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} < \eta \quad \text{for } {}^\exists x \in D\right\}$$

Proposition 4.2 (Dichotomy of global capacity density)

Let $\phi(r) = \inf_{x \in \mathbb{R}^n} \dfrac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \cap E)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))}$ *for* $E \subset \mathbb{R}^n$*. Then*

$$\lim_{r\to\infty} \phi(r) = \begin{cases} 0 & \text{if } \phi(r) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \phi(r) > 0 \text{ for some } r > 0. \end{cases}$$

Proof. $\widetilde{\mathrm{Inr}}_\eta(D) \le \widetilde{\mathrm{Inr}}_{\eta'}(D) \le A\widetilde{\mathrm{Inr}}_\eta(D)$ $(0 < \eta < \eta' < 1) \implies$ dichotomy .

Let $D = \mathbb{R}^n \setminus E$. Suppose $\phi(r) \ge \eta > 0$ for some $\eta > 0$ and $r > 0$. Then $\widetilde{\mathrm{Inr}}_\eta(D) = w_\eta(D) \le r$. Take $\eta < {}^\forall\eta' < 1$. Then $\widetilde{\mathrm{Inr}}_{\eta'}(D) \le Ar$. Hence if $R > Ar$, then for ${}^\forall x$

$$\frac{\mathrm{Cap}_{B(x,2R)}(B(x,R) \cap E)}{\mathrm{Cap}_{B(x,2R)}(B(x,R))} \ge \eta' \implies \liminf_{R\to\infty} \phi(R) \ge {}^\forall\eta' \implies \lim_{R\to\infty} \phi(R) = 1$$

as $\widetilde{\mathrm{Inr}}_{\eta'}(D) = \sup\left\{R > 0 : \dfrac{\mathrm{Cap}_{B(x,2R)}(B(x,R) \setminus D)}{\mathrm{Cap}_{B(x,2R)}(B(x,R))} < \eta' \quad \text{for } {}^\exists x \in D\right\}$. □

Dichotomy of global density of general capacities

Let $1 < p < \infty$ and let w be a p-admissible weight as in [HKM93, Chapter 1]. Note an A_p-weight is admissible. Write $d\mu(x) = w(x)dx$. Let $K \subset D \subset \mathbb{R}^n$. Define

$$\mathrm{Cap}_{p,\mu}(K, D) = \inf\left\{ \int_D |\nabla u|^p d\mu : \ u \ge 1 \text{ on } K, u \in C_0^\infty(D) \right\}.$$

Theorem 5.1 (Dichotomy of weighted L^p-capacity [AI15])

Let $\lambda > 1$. For $E \subset \mathbb{R}^n$ and $r > 0$ define

$$\mathcal{D}_{B,\lambda B}(\mathrm{Cap}_{p,\mu}, E, r) = \inf_{x \in \mathbb{R}^n} \frac{\mathrm{Cap}_{p,\mu}(E \cap B(x,r), B(x,\lambda r))}{\mathrm{Cap}_{p,\mu}(B(x,r), B(x,\lambda r))}.$$

Then

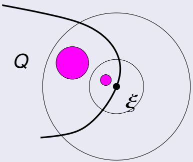
$$\lim_{r\to\infty} \mathcal{D}_{B,\lambda B}(\mathrm{Cap}_{p,\mu}, E, r) = \begin{cases} 0 & \text{if } \mathcal{D}_{B,\lambda B}(\mathrm{Cap}_{p,\mu}, E, r) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \mathcal{D}_{B,\lambda B}(\mathrm{Cap}_{p,\mu}, E, r) > 0 \text{ for some } r > 0. \end{cases}$$

Density can be taken over a general open set. Write $Q(x,r) = \{ry + x \in \mathbb{R}^n : y \in Q\}$ for $Q \subset \mathbb{R}^n$, $r > 0$ and $x \in \mathbb{R}^n$.

Theorem 5.2 (Density on general open sets [AI15])

Let $\overline{Q} \subset Q^*$. Suppose Q satisfies the interior corkscrew condition, i.e., there exist $0 < \kappa < 1$ and $\rho_0 > 0$ such that

$$\xi \in \partial Q,\ 0 < r \le \rho_0 \implies B(\xi, r) \cap Q \supset {}^{\exists}B(y, \kappa r).$$



For $E \subset \mathbb{R}^n$ and $r > 0$ define

$$\mathcal{D}_{Q,Q^*}(\mathrm{Cap}_{p,\mu}, E, r) = \inf_{x \in \mathbb{R}^n} \frac{\mathrm{Cap}_{p,\mu}(E \cap Q(x,r), Q^*(x,r))}{\mathrm{Cap}_{p,\mu}(Q(x,r), Q^*(x,r))}.$$

Then

$$\lim_{r \to \infty} \mathcal{D}_{Q,Q^*}(\mathrm{Cap}_{p,\mu}, E, r) = \begin{cases} 0 & \text{if } \mathcal{D}_{Q,Q^*}(\mathrm{Cap}_{p,\mu}, E, r) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \mathcal{D}_{Q,Q^*}(\mathrm{Cap}_{p,\mu}, E, r) > 0 \text{ for some } r > 0. \end{cases}$$

Let (X, d, μ): metric (doubling)measure space with p-Poincaré inequality, $1 < p < \infty$. Define the inner metric d_{in} by $d_{\text{in}}(x, y) = \inf_{\gamma_{xy}} \ell(\gamma_{xy})$. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ and $B_{\text{in}}(x, r) = \{y \in X : d_{\text{in}}(x, y) < r\}$. For $E \subset X$ and $\tau > 1$, define

$$\mathcal{D}(r, \tau, E) = \inf_{x \in X} \frac{\text{Cap}_p(E \cap B(x, r), B(x, \tau r))}{\text{Cap}_p(B(x, r), B(x, \tau r))},$$

$$\mathcal{D}_{\text{in}}(r, \tau, E) = \inf_{x \in X} \frac{\text{Cap}_p(E \cap B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r))}{\text{Cap}_p(B_{\text{in}}(x, r), B_{\text{in}}(x, \tau r))}.$$

Theorem 5.3 (Dichotomy of variational capacity in MMS [ABBS17])

- $\mathcal{D}_{\text{in}}(r, \tau, E)$ *enjoys the dichotomy:*

$$\lim_{r \to \infty} \mathcal{D}_{\text{in}}(r, \tau, E) = \begin{cases} 0 & \text{if } \mathcal{D}_{\text{in}}(r, \tau, E) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \mathcal{D}_{\text{in}}(r, \tau, E) > 0 \text{ for some } r > 0. \end{cases}$$

- $\mathcal{D}(r, \tau, E)$ *fails the dichotomy:* $^{\exists}(X, d, \mu)$ *supporting a* 1*-Poincaré inequality and* $E \subset X$ *such that* $0 < \liminf_{r \to \infty} \mathcal{D}(r, \tau, E) < 1$ *for* $^{\forall}\tau > 1$.

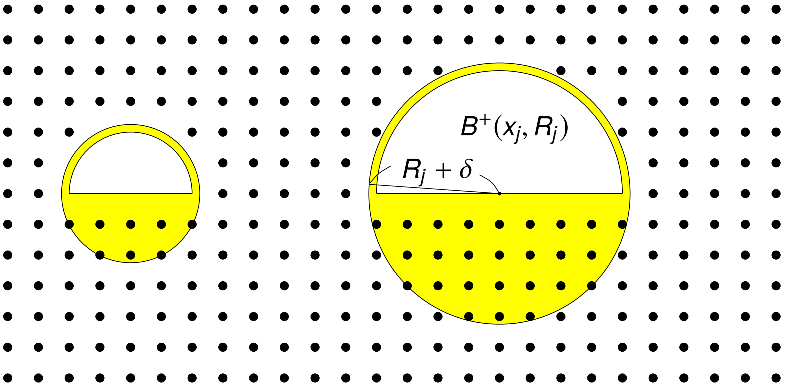
Let $X = \mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} B^+(x_j, R_j)$ with $x_j = (4^j, 0, \dots, 0)$ and $R_j = 2^j$, $j = 1, 2, \dots$.

- 1-Poincaré inequality ◌ X is the closure of a uniform domain.

No dichotomy w.r.t. $B_X(x, r) = \{y \in X : d(x, y) < r\}$ for

$$E = \bigcup_{z \in \mathbb{Z}^n \setminus H} B(z, \delta), \quad \text{where } 0 < \delta \le \frac{1}{4}, \quad H = \bigcup_{j=1}^{\infty} B^+(x_j, R_j)\left[\frac{1}{2}\right]$$

and $B^+(x_j, R_j)\left[\frac{1}{2}\right]$ is the $\frac{1}{2}$-neighborhood of $B^+(x_j, R_j)$.



Dichotomy of global density of Riesz capacity

Let $0 < \alpha < n$. Define the Riesz capacity by

$$C_\alpha(E) = \inf\{\|\mu\| : U_\alpha^\mu \geq 1 \text{ on } E\}, \quad \text{where } U_\alpha^\mu(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} d\mu(y).$$

Theorem 5.4 (Dichotomy of global density of Riesz capacity [Aik16])

Suppose a bounded open set Ω satisfies the interior corkscrew condition. Define $\mathcal{D}_\Omega(C_\alpha, E, r) = \inf_{x \in \mathbb{R}^n} \dfrac{C_\alpha(E \cap \Omega(x,r))}{C_\alpha(\Omega(x,r))}$. *If $0 < \alpha \leq 2$, then*

$$\lim_{r\to\infty} \mathcal{D}_\Omega(C_\alpha, E, r) = \begin{cases} 0 & \text{if } \mathcal{D}_\Omega(C_\alpha, E, r) = 0 \text{ for all } r > 0, \\ 1 & \text{if } \mathcal{D}_\Omega(C_\alpha, E, r) > 0 \text{ for some } r > 0. \end{cases}$$

Remark 5.5

With the aid of Choquet's des cages de Faraday grillagées or the grounded Faraday cage ([Cho75]), we have extended the range α to $0 < \alpha < n$.

Counterexample

Remark 6.1 (No dichotomy for Sobolev capacity [ABBS17, Example 7.2])

The Sobolev capacity in $\mathbb{R}^n$

$$C_p(E) = \inf\{\|u\|_p^p + \|\nabla u\|_p^p : u \ge 1 \text{ on } E,\ u \in C_0^\infty(\mathbb{R}^n)\}$$

has no dichotomy, i.e., for bounded open ${}^\forall\Omega$, Borel ${}^\exists E \subset \mathbb{R}^n$ such that

$$0 < \liminf_{r\to\infty} \mathcal{D}_\Omega(C_p, E, r) \le \limsup_{r\to\infty} \mathcal{D}_\Omega(C_p, E, r) < 1.$$

Similarly, if the convolution kernel $k \in L^1(\mathbb{R}^n)$, then

$$C_k(E) = \inf\{\|\mu\| : k * \mu \ge 1 \text{ on } E\}$$

has no dichotomy, i.e., for bounded open ${}^\forall\Omega$, Borel ${}^\exists E \subset \mathbb{R}^n$ such that

$$0 < \liminf_{r\to\infty} \mathcal{D}_\Omega(C_k, E, r) \le \limsup_{r\to\infty} \mathcal{D}_\Omega(C_k, E, r) < 1.$$

No dichotomy for C_k with $k \in L^1(\mathbb{R}^n)$

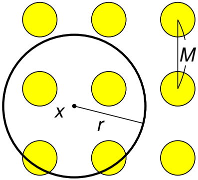
If $\|k\|_1 < \infty$, then $\|k * \chi_E\|_\infty \le \|k\|_1 < \infty$, and by definition

(1) $$C_k(E) \ge \frac{|E|}{\|k\|_1} \quad \text{for every Borel set } E.$$

If $r \ge 1$, then $k * \chi_{B(x,2r)} \ge \int_{|y|\le 1} k(y)dy > 0$ on $B(x,r)$, so that

(2) $$C_k(B(x,r)) \le \|\chi_{B(x,2r)}\|_1 \Big/ \int_{|y|\le 1} k(y)dy = A_1|B(x,r)|.$$

Let $M > 4$ and $E = \bigcup_{z\in(M\mathbb{Z})^n} B(z,1)$. If $r > M$, then $\#\{B(z,1) : B(z,1)\cap B(x,r) \ne \emptyset\} \le A_2(r/M)^n$, $|E\cap B(x,r)|/|B(x,r)| \ge A_2 M^{-n}$. By (1), (2) and the subadditivity of C_k



$$A^{-1}M^{-n} \le \frac{C_k(E\cap B(x,r))}{C_k(B(x,r))} \le \frac{A_2(r/M)^n C_k(B(0,1))}{|B(x,r)|/\|k\|_1} \le AM^{-n}.$$

Hence, if M is large, then

$$0 < \liminf_{r\to\infty} \mathcal{D}(C_k, E, r) \le \limsup_{r\to\infty} \mathcal{D}(C_k, E, r) < 1.$$

References I

[ABBS17] H. Aikawa, A. Björn, J. Björn, and N. Shanmugalingam, Adv. Calc. Var. (2017), to appear.

[AI15] H. Aikawa and T. Itoh, Proc. Amer. Math. Soc. **143** (2015), no. 12, 5381–5393.

[Aik98] H. Aikawa, Math. Ann. **312** (1998), no. 2, 289–318.

[Aik15] ______, Rev. Mat. Iberoam. **31** (2015), no. 3, 1041–1106.

[Aik16] ______, Studia Math. **232** (2016), no. 3, 267–278.

[AM96] H. Aikawa and M. Murata, J. Anal. Math. **69** (1996), 137–152.

[Anc86] A. Ancona, J. London Math. Soc. (2) **34** (1986), no. 2, 274–290.

[Bañ92] R. Bañuelos, Partial differential equations with minimal smoothness and applications (Chicago, IL, 1990), IMA Vol. Math. Appl., vol. 42, Springer, New York, 1992, pp. 37–48.

References II

[BnC94] R. Bañuelos and T. Carroll, Duke Math. J. **75** (1994), no. 3, 575–602.

[BØ87] R. Bañuelos and B. Øksendal, Proc. Edinburgh Math. Soc. (2) **30** (1987), no. 2, 273–287.

[Cho75] G. Choquet, Bull. Sci. Math. (2) **99** (1975), no. 3, 157–164.

[Chu84] K. L. Chung, Ann. Inst. H. Poincaré Probab. Statist. **20** (1984), no. 4, 349–351.

[CM83] M. Cranston and T. R. McConnell, Z. Wahrsch. Verw. Gebiete **65** (1983), no. 1, 1–11.

[DS84] E. B. Davies and B. Simon, J. Funct. Anal. **59** (1984), no. 2, 335–395.

[Hay78] W. K. Hayman, Applicable Anal. **7** (1977/78), no. 3, 247–254.

References III

[HKM93] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications.

[HP78] W. K. Hayman and C. Pommerenke, Bull. London Math. Soc. **10** (1978), no. 2, 219–224.

[MS05] V. Maz′ya and M. Shubin, Lett. Math. Phys. **74** (2005), no. 2, 135–151.

[Per60] A. Persson, Math. Scand. **8** (1960), 143–153.

[Ste80] D. A. Stegenga, Michigan Math. J. **27** (1980), no. 2, 247–252.

[vdB12] M. van den Berg, Potential Anal. **36** (2012), no. 4, 607–616.

[vdBC09] M. van den Berg and T. Carroll, Bull. Lond. Math. Soc. **41** (2009), no. 6, 980–986.