On the generator of a killed Feller process

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(based on a joint work with B. Baeumer
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Probability and Analysis

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Motivation

Consider the following Cauchy problem

$$\begin{aligned} \partial_t u(x,t) &= \mathcal{L}u(x,t) \quad \forall\, x \in \Omega,\ t > 0; \\ u(x,0) &= f(x) \quad \forall\, x \in \Omega; \\ u(x,t) &= 0 \quad \forall\, x \notin \Omega, t \geq 0, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^d$ is a bounded domain and $\mathcal{L}$ is some (pseudo-)differential operator acting on u in x.

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Idea: Find a stronlgy continuous semigroup $\{P_t^\Omega, t \geq 0\}$ whose infinitesimal generator equals $\mathcal{L}$ and take $u = P_t^\Omega f$.

Motivation

If $\{P_t^{\Omega}, t \geq 0\}$ is uniformly bounded, this approach would also provide a solution to the fractional Cauchy problem

$$\begin{aligned}
\partial_t^{\beta} v(x,t) &= \mathcal{L} v(x,t) \quad \forall\, x \in \Omega,\ \ t > 0;\\
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where ∂_t^{β} is the Caputo fractional derivative of order $0 < \beta < 1$.

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where ∂_t^β is the Caputo fractional derivative of order $0 < \beta < 1$. Namely,

$$v(x,t) = \int_0^\infty g_\beta(s) P^\Omega_{(t/s)^\beta} f(x) ds,$$

where g_β denotes the probability density function of the standard stable subordinator, with Laplace transform $\int_0^\infty e^{-st} g_\beta(t)dt = e^{-s^\beta}$ (B.Baeumer, M.Meerschaert, 2001).

Generator of a Feller process

Definition

$(X_t)_{t\geq 0}$ is a Feller process on $\mathbb{R}^d$, when

$$P_t f(x) := \mathbb{E}^x f(X_t)$$

defines a strongly continuous contraction semigroup on $C_0(\mathbb{R}^d)$.

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$$L^\sharp f(x) := \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}, \quad x \in \mathbb{R}^d$$ – pointwise formula for L

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Fact: If $f \in C_0(\mathbb{R}^d)$, $L^\sharp f(x)$ exists for all $x \in \mathbb{R}^d$ and $L^\sharp f \in C_0(\mathbb{R}^d)$, then $f \in \mathcal{D}(L)$.

Generator of a Feller process

Courrège, von Waldenfels 65': If $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(L)$, then for $f \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} Lf(x) = &- c(x)f(x) + l(x) \cdot \nabla f(x) + \nabla(Q(x)\nabla f(x)) \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} (f(x+y) - f(x) - \nabla f(x) \cdot y 1_B(y)) N(x, dy) \\ =&: PDO[f](x) \end{aligned}$$

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for some $c(x) \geq 0$, $l(x) \in \mathbb{R}^d$, $Q(x) \in \mathbb{R}^{d \times d}$ symmetric and positive semidefinite, B the unit ball, and $N(x, \cdot)$ a positive measure satisfying

$$\int_{\mathbb{R}^d \setminus \{0\}} \min(|y|^2, 1) N(x, dy) < \infty.$$

Killed Feller process

$\Omega \subseteq \mathbb{R}^d$ – bounded domain

$C_0(\Omega) := \{f : \Omega \to \mathbb{R} : f \text{ continuous and } f(x) \to 0 \text{ as } x \to \partial\Omega\}$

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$$X_t^\Omega := \begin{cases} X_t, & t < \tau_\Omega \\ \partial, & t \geq \tau_\Omega \end{cases}$$

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Definition

$x \in \partial\Omega$ is regular for Ω :$\iff$ $\mathbb{P}^x(\tau_\Omega = 0) = 1$.

Ω is regular :$\iff$ all $x \in \partial\Omega$ are regular.

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Definition

$\{P_t, t \geq 0\}$ is strong Feller $:\iff$ $P_t f \in C_b(\mathbb{R}^d)$ for all $t > 0$ and all f measurable bounded with compact support in $\mathbb{R}^d$.

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X_t *is doubly Feller* $:\iff$ X_t *is Feller and* $\{P_t, t \geq 0\}$ *is doubly Feller.*

Chung 86': X_t doubly Feller, Ω regular $\implies$

$$P_t^\Omega f(x) := \mathbb{E}^x f(X_t^\Omega), \quad x \in \Omega, t \geq 0$$

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L_Ω – the infinitesimal generator of $\{P_t^\Omega, t \geq 0\}$ with domain $\mathcal{D}(L_\Omega)$

Main results

Theorem (BB,TL,MM)

Suppose X_t is doubly Feller on $\mathbb{R}^d$ and $\Omega \subseteq \mathbb{R}^d$ is regular. Then

$$\mathcal{D}(L_\Omega) = \{f \in C_0(\Omega) : L^\sharp f \in C_0(\Omega)\}.$$

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Also, for all $f \in D(L_\Omega)$ and $x \in \Omega$ we have $L_\Omega f(x) = L^\sharp f(x)$ and $\dfrac{P_t f - f}{t} \to L^\sharp f$ uniformly on compacta in Ω.

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Examples:

- If X_t is a Brownian motion in $\mathbb{R}^d$ and $\Omega \subseteq \mathbb{R}^d$ is regular, then $C_c^2(\Omega) \subseteq \mathcal{D}(L_\Omega)$ and $L_\Omega f = PDO[f] = \frac{1}{2}\Delta f$ for $f \in C_c^2(\Omega)$.

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- If X_t is a rotationally invariant α-stable Lévy process in $\mathbb{R}^d$ with $0 < \alpha < 2$, then $Lf = PDO[f] = -(-\Delta)^{\alpha/2} f$ for $f \in C_0^2(\mathbb{R}^d)$, but $C_c^\infty(\Omega) \not\subset \mathcal{D}(L_\Omega)$.

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Theorem (BB,TL,MM)

Suppose X_t is doubly Feller on $\mathbb{R}^d$ and $\Omega \subseteq \mathbb{R}^d$ is regular. Then

$$\mathcal{D}(L_\Omega) = \Big\{ f \in C_0(\Omega) : \exists g \in C_0(\Omega), (f_n) \subseteq \mathcal{D}(L) \text{ such that } f_n \to f \text{ in } C_0(\mathbb{R}^d) \text{ and } Lf_n \to g \text{ unif. on compacta in } \Omega \Big\},$$

and for f, g as above we have $L_\Omega f = g$.

Main results

$$C_0^2(\Omega) := C_0(\Omega) \cap C^2(\Omega)$$

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- *For every $f \in \mathcal{D}(L_\Omega)$ there exists $(f_n) \subseteq C_0^2(\Omega)$ such that $f_n \to f$ uniformly in $C_0(\Omega)$ and $PDO[f_n] \to L_\Omega f$ uniformly on compacta in Ω.*

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In particular, if $f \in C_0^2(\Omega)$ and $PDO[f] \in C_0(\Omega)$, then $f \in \mathcal{D}(L_\Omega)$ and $L_\Omega f(x) = PDO[f](x)$ for every $x \in \Omega$.

Proofs

For $f \in C_0(\Omega)$ we consider

$$\frac{P_t f(x) - f(x)}{t} - \frac{P_t^{\Omega} f(x) - f(x)}{t} = \frac{\mathbb{E}^x[f(X_t) 1_{\{\tau_\Omega < t\}}]}{t}.$$

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Strong continuity and strong Markov property of $P_t \implies$

$$\frac{|\mathbb{E}^x[f(X_t) 1_{\{\tau_\Omega < t\}}]|}{t} \leq \frac{\varepsilon \mathbb{P}^x(\tau_\Omega < t)}{t}$$

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Let $U \subset\subset \Omega$ and choose $r > 0$ such that $B(x,r) \subset \Omega \quad \forall x \in U$.

R.Schilling, J.Wang, 2012:

$\exists M > 0$: $\dfrac{\mathbb{P}^x(\tau_{B(x,r)} < t)}{t} \leq M \quad \forall x \in U, t > 0.$

Proofs

Suppose $f \in C_0(\Omega)$ and that

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as $t \to 0$ for some $g \in C_0(\Omega)$, for all $x \in \Omega$.

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The resolvent $(\lambda - L_\Omega)^{-1}$ exists for all $\lambda > 0$, and maps $C_0(\Omega)$ onto $\mathcal{D}(L_\Omega) \implies \exists h \in \mathcal{D}(L_\Omega)$ such that $(I - L_\Omega)h = f - g$.

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By previous calculation applied to h,

$$L_\Omega h(x) - g(x) = \lim_{t\to 0} \frac{P_t h(x) - h(x) - (P_t f(x) - f(x))}{t}, \quad x \in \Omega.$$

Proofs

Hence, for $u = h - f$ we get

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Since P_t is a contraction, $P_t u(x_0) \leq \|P_t u\| \leq \|u\| = u(x_0)$ and therefore

$$0 \geq \frac{P_t u(x_0) - u(x_0)}{t} \to u(x_0) > 0$$

as $t \to 0$, which is a contradiction. Hence $\sup_{x \in \Omega} |u(x)| = 0$ and therefore $h = f$.

Proofs

Next we show that $f \in \mathcal{D}(L_\Omega)$ can be approximated locally in the graph norm by functions in $\mathcal{D}(L)$, namely by the functions

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As $P_t f$ is continuous in t and $\|P_t f\| \leq \|f\|$, it is not hard to check that $f_\lambda = \lambda \int_0^\infty e^{-\lambda t} P_t f \, dt$ and

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in $C_0(\mathbb{R}^d)$. Furthermore, $f_\lambda \in \mathcal{D}(L)$ and by definition,

$$L f_\lambda = \lambda f_\lambda - \lambda f.$$

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$$\begin{aligned}
\lim_{\lambda\to\infty} Lf_\lambda(x) &= \lim_{\lambda\to\infty} \lambda^2 \int_0^\infty e^{-\lambda t} P_t f(x)\, dt - \lambda f(x) \\
&= \lim_{\lambda\to\infty} \int_0^\infty u e^{-u} \frac{P_{(u/\lambda)} f(x) - f(x)}{(u/\lambda)}\, du \\
&= L_\Omega f(x)
\end{aligned}$$

uniformly in $x \in U$.

Proofs

Reverse set inclusion: suppose $f \in C_0(\Omega)$ and for some $(f_n) \subseteq \mathcal{D}(L)$ we have $f_n \to f$ in $C_0(\mathbb{R}^d)$ and $Lf_n(x) \to g(x)$ uniformly in $x \in U \subset\subset \Omega$ for some $g \in C_0(\Omega)$.

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Let $h = (I - L_\Omega)^{-1}(f - g)$ so that

$$h - f = L_\Omega h - g.$$

The resolvent maps $C_0(\Omega)$ onto $\mathcal{D}(L_\Omega) \implies$

$$\exists h_n \in \mathcal{D}(L) : h_n \to h \text{ in } C_0(\mathbb{R}^d) \text{ and } Lh_n(x) \to L_\Omega h(x) \quad \forall x \in \Omega,$$

uniformly on compacta, by what we have already proven.

Proofs

Let $u = h - f$ and assume $u(x_0) = \|u\| > \epsilon$ for some $\epsilon > 0$. Let $u_n = h_n - f_n$ so that

$$Lu_n(x) \to L_\Omega h(x) - g(x) = u(x)$$

uniformly in $x \in U \subset\subset \Omega$.

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u_n converges uniformly to $u \implies \exists N > 0, U \subset\subset \Omega$ such that

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u_n converges uniformly to $u \implies \exists N > 0, U \subset\subset \Omega$ such that

$$\{x_n : u_n(x_n) = \|u_n\|\} \subset U \qquad \forall n > N.$$

$u_n(x_n) > \epsilon/2$ for large n and $Lu_n(x_n) \leq 0$ by PMP $\implies$ $u_n(x) - Lu_n(x)$ cannot converge uniformly on U to 0.

This is a contradiction, and hence $u \equiv 0 \implies h = f \in \mathcal{D}(L_\Omega)$.

Example: Application to stable processes

The positive and negative Riemann-Liouville fractional derivatives are defined by

$$\mathbb{D}^{\alpha}_{[L,x]} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_L^x f(y)(x-y)^{n-\alpha-1} dy,$$
$$\mathbb{D}^{\alpha}_{[x,R]} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^R f(y)(y-x)^{n-\alpha-1} dy$$

for any non-integer $\alpha > 0$ and any $-\infty \le L < x < R \le \infty$, where $n - 1 < \alpha < n$.

Theorem (BB,TL,MM)

Let X_t be any stable Lévy process with index $1 < \alpha < 2$ and let $\Omega = (L, R)$. Then for all $x \in \Omega$ and any $f \in C_0^2(\Omega)$ such that $PDO[f] \in C_0(\Omega)$ we have

$$L_\Omega f(x) = -af'(x) + b\,\mathbb{D}^{\alpha}_{[L,x]} f(x) + c\,\mathbb{D}^{\alpha}_{[x,R]} f(x).$$

THANK YOU FOR YOUR ATTENTION !