Invariant, super and quasi-martingale functions of a Markov process

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- E : a Lusin topological space endowed with the Borel σ-algebra $\mathcal{B}$
- $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x, \zeta)$ be a right Markov process with state space E, transition function :
- $(P_t)_{t\geqslant 0}$: the transition function of X,

$$P_t u(x) = \mathbb{E}^x(u(X_t); t < \zeta), t \geqslant 0, x \in E.$$

Proposition

The following assertions are equivalent for a non-negative real-valued $\mathcal{B}$-measurable function u and $\beta \geqslant 0$.
(i) $(e^{-\beta t}u(X_t))_{t\geqslant 0}$ is a right continuous $\mathcal{F}_t$-supermartingale w.r.t. $\mathbb{P}^x$ for all $x \in E$.
(ii) The function u is β-excessive.

First aim: To show that this connection can be extended between the space of **differences of excessive functions** on the one hand, and **quasimartingales** on the other hand, with concrete applications to semi-Dirichlet forms.

Supermedian and excessive functions

- For $\beta \geqslant 0$, a $\mathcal{B}$-measurable function $f : E \to [0, \infty]$ is called β**-supermedian** if $P_t^\beta f \leqslant f$, $t \geqslant 0$;
$(P_t^\beta)_{t \geqslant 0}$ denotes the β-level of the semigroup of kernels $(P_t)_{t \geqslant 0}$, $P_t^\beta := e^{-\beta} P_t$.
- If f is β-supermedian and $\lim\limits_{t \to 0} P_t f = f$ pointwise on E, then it is called β**-excessive**.
- A $\mathcal{B}$-measurable function f is β-excessive if and only if $\alpha U_{\alpha+\beta} f \leqslant f$, $\alpha > 0$, and $\lim\limits_{\alpha \to \infty} \alpha U_\alpha f = f$ pointwise on E,

where $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ is the resolvent family of the process X, $U_\alpha := \int_0^\infty e^{-\alpha t} P_t dt$.

- $\mathcal{U}_\beta$:= the β-level of the resolvent $\mathcal{U}$, $\mathcal{U}_\beta := (U_{\beta+\alpha})_{\alpha > 0}$;
- $E(\mathcal{U}_\beta)$:= the convex cone of all β-excessive functions.

If $\beta = 0$ we drop the index β from notations.

Proof. (i) $\Longrightarrow$ (ii). If $(e^{-\beta t}u(X_t))_{t\geqslant 0}$ is a right-continuous supermartingale then by taking expectations we get that $e^{-\beta t}\mathbb{E}^x u(X_t) \leqslant \mathbb{E}^x u(X_0)$, hence u is β-supermedian.
- If u is β-supermedian then to prove that it is β-excessive reduces to prove that u is finely continuous, which in turns follows by the well known characterization for the fine continuity:
u is finely continuous if and only if $u(X)$ has right continuous trajectories $\mathbb{P}^x$-a.s. for all $x \in E$.

(ii) $\Longrightarrow$ (i). Since u is β-supermedian and by the Markov property we have for all $0 \leqslant s \leqslant t$

$$\mathbb{E}^x[e^{-\beta(t+s)}u(X_{t+s})|\mathcal{F}_s] = e^{-\beta(t+s)}\mathbb{E}^{X_s}u(X_t) =$$

$$e^{-\beta(t+s)}P_t u(X_s) \leqslant e^{-\beta s}u(X_s),$$

hence $(e^{-\beta t}u(X_t))_{t\geqslant 0}$ is an $\mathcal{F}_t$-supermartingale.
The right-continuity of the trajectories follows by the fine continuity of u via the previously mentioned characterization.

I. Differences of excessive functions and quasimartingales of Markov processes

Theorem. *The following assertions are equivalent for a non-negative real-valued $\mathcal{B}$-measurable function u.*

(i) $u(X)$ is an $\mathcal{F}_t$-semimartingale w.r.t. all $\mathbb{P}^x$, $x \in E$.

(ii) u is locally the difference of two finite 1-excessive functions.

[E. Çinlar, J. Jacod, P. Protter, M.J. Sharpe, *Z. W. verw. Gebiete* 1980]

Quasimartingales

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses.

An $\mathcal{F}_t$-adapted, right-continuous integrable process $(Z_t)_{t \geqslant 0}$ is called $\mathbb{P}$-**quasimartingale** if

$$Var^{\mathbb{P}}(Z) := \sup_{\tau} \mathbb{E}\{\sum_{i=1}^{n} |\mathbb{E}[Z_{t_i} - Z_{t_{i-1}} | \mathcal{F}_{t_{i-1}}]| + |Z_{t_n}|\} < \infty,$$

where the supremum is taken over all partitions $\tau : 0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n < \infty$.

M. Rao's characterization of the quasimartingales

A real-valued process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *satisfying the usual hypotheses is a quasimartingale if and only if it is the difference of two positive right-continuous* $\mathcal{F}_t$*-supermartingales.*

[P.E. Protter, *Stochastic Integration and Diff. Equations.* Springer 2005]

Remark. If $u(X)$ is a quasimartingale, then the following two conditions for u are necessary:

(i) $\sup_{t>0} P_t|u| < \infty$

and

(ii) u is finely continuous.

The first assertion is clear since for each $x \in E$

$$\sup_t P_t|u|(x) = \sup_t \mathbb{E}^x|u(X_t)| \leqslant Var^{\mathbb{P}^x}(u(X)) < \infty.$$

The second one follows from the Blumenthal-Getoor's characterization of the fine continuity.

For a real-valued function u,
a finite partition τ of $\mathbb{R}^+$, $\tau : 0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n < \infty$,
and $\alpha > 0$ we set

$$V_\tau^\alpha(u) := \sum_{i=1}^{n} P_{t_{i-1}}^\alpha |u - P_{t_i - t_{i-1}}^\alpha u| + P_{t_n}^\alpha |u|,$$

$$V^\alpha(u) := \sup_\tau V_\tau^\alpha(u).$$

where the supremum is taken over all finite partitions of $\mathbb{R}_+$.

Admissible sequence of partitions: an increasing sequence $(\tau_n)_{n\geqslant 1}$ of finite partitions of $\mathbb{R}_+$ such that $\bigcup_{k\geqslant 1} \tau_k$ is dense in $\mathbb{R}_+$ and if $r \in \bigcup_{k\geqslant 1} \tau_k$ then $r + \tau_n \subset \bigcup_{k\geqslant 1} \tau_k$ for all $n \geqslant 1$.

Theorem

Let u be a real-valued $\mathcal{B}$*-measurable function and* $\beta \geqslant 0$ *such that* $P_t|u| < \infty$ *for all* t*. Then the following assertions are equivalent.*

(i) $(e^{-\beta t}u(X_t))_{t\geqslant 0}$ *is a* $\mathbb{P}^x$*-quasimartingale for all* $x \in E$*.*

(ii) u is finely continuous and $\sup_n V^{\beta}_{\tau_n}(u) < \infty$ *on E for one (hence all) admissible sequence of partitions* $(\tau_n)_n$*.*

(iii) u is a difference of two real-valued β*-excessive functions.*

[L. Beznea, I. Cîmpean, *Trans. Amer. Math. Soc.* 2017]

Comments about the proof

- Key idea: By the Markov property one can show that

$Var^{\mathbb{P}^x}((e^{-\alpha t}u(X_t)_{t\geqslant 0}) = V^\alpha(u)(x)$ for all $x \in E$,

meaning that assertion (i) holds if and only if $V^\alpha(u) < \infty$.

- $V^\alpha(u)$ is a supremum of measurable functions taken over an uncountable set of partitions, hence it may no longer be measurable. However, the set $[V^\alpha(u) < \infty]$ is of interest to us, not necessarily $V^\alpha(u)$.

- It turns out that $[V^\alpha(u) < \infty]$ is measurable and, moreover, it is completely determined by $\sup\limits_n V^\alpha_{\tau_n}(u)$ for any admissible sequence of partitions $(\tau_n)_{n\geqslant 1}$. This aspect is crucial in order to give criteria to check the quasimartingale nature of $u(X)$.

Criteria for quasimartingale functions on L^p-spaces

Assume that μ is a σ-finite **sub-invariant measure** for $(P_t)_{t\geqslant 0}$;
i.e., $\mu \circ P_t \leqslant \mu$ for all $t > 0$.

Proposition

The following assertions are equivalent for a $\mathcal{B}$-measurable function $u \in \bigcup_{1\leqslant p\leqslant\infty} L^p(\mu)$ and $\beta \geqslant 0$.

(i) There exists a μ-version $\widetilde{u}$ of u such that $(e^{-\beta t}\widetilde{u}(X_t))_{t\geqslant 0}$ is a $\mathbb{P}^x$-quasimartingale for $x \in E$ μ-a.e.

(ii) For an admissible sequence of partitions of $(\tau_n)_{n\geqslant 1}$ of $\mathbb{R}_+$, $\sup_n V^{\beta}_{\tau_n}(u) < \infty$ μ-a.e.

(iii) There exist $u_1, u_2 \in E(\mathcal{U}_\beta)$ finite μ-a.e. such that $u = u_1 - u_2$ μ-a.e.

Remark. If u is finely continuous and one of the above equivalent assertions is satisfied then all of the statements hold quasi everywhere, not only μ-a.e., since an μ-negligible finely open set is μ-polar.
If in addition μ is a reference measure then the assertions hold everywhere on E.

The generator on L^p-spaces

Since μ is sub-invariant, $(P_t)_{t\geqslant 0}$ and $\mathcal{U}$ extend to strongly continuous semigroup resp. resolvent family of contractions on $L^p(\mu)$, $1 \leqslant p < \infty$.

- The corresponding **generator** $(\mathsf{L}_p, D(\mathsf{L}_p) \subset L^p(\mu))$ is defined as

$$D(\mathsf{L}_p) = \{U_\alpha f : f \in L^p(m)\},$$

$$\mathsf{L}_p(U_\alpha f) := \alpha U_\alpha f - f \quad \text{for all } f \in L^p(\mu),\ 1 \leqslant p < \infty,$$

with the remark that this definition is independent of $\alpha > 0$.

- The analogous notations for the **dual** structure are $\widehat{P}_t$ and $(\widehat{\mathsf{L}}_p, D(\widehat{\mathsf{L}}_p))$, and note that the adjoint of L_p is $\widehat{\mathsf{L}}_{p^*}$; $\frac{1}{p} + \frac{1}{p^*} = 1$.

We focus our attention on a class of β-quasimartingale functions which arises as a natural extension of $D(\mathsf{L}_p)$.

- Any function $u \in D(\mathsf{L}_p)$, $1 \leqslant p < \infty$, has a representation $u = U_\beta f = U_\beta(f^+) - U_\beta(f^-)$ with $U_\beta(f^\pm) \in E(\mathcal{U}_\beta) \cap L^p(\mu)$, hence u has a β-quasimartingale version for all $\beta > 0$; moreover, $\|P_t u - u\|_p = \left\| \int_0^t P_s \mathsf{L}_p u ds \right\|_p \leqslant t \|\mathsf{L}_p u\|_p$.

- The converse is also true, namely if $1 < p < \infty$, $u \in L^p(\mu)$, and $\|P_t u - u\|_p \leqslant const \cdot t$, $t \geqslant 0$, then $u \in D(\mathsf{L}_p)$. But this is no longer the case if $p = 1$ (because of the lack of reflexivity of L^1), i.e. $\|P_t u - u\|_1 \leqslant const \cdot t$ does not imply $u \in D(\mathsf{L}_1)$.

However, it turns out that this last condition on $L^1(m)$ is yet enough to ensure that u is a β-quasimartingale function.

Proposition

Let $1 \leqslant p < \infty$ and suppose $\mathcal{A} \subset \{u \in L_+^{p^*}(m) : \|u\|_{p^*} \leqslant 1\}$, $\widehat{P}_s \mathcal{A} \subset \mathcal{A}$ for all $s \geqslant 0$, and $E = \bigcup_{f \in \mathcal{A}} \operatorname{supp}(f)$ m-a.e. If $u \in L^p(m)$ satisfies

$$\sup_{f \in \mathcal{A}} \int_E |P_t u - u| f dm \leqslant const \cdot t \text{ for all } t \geqslant 0,$$

then there exists an m-version $\widetilde{u}$ of u such that $(e^{-\beta t} \widetilde{u}(X_t))_{t \geqslant 0}$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$ m-a.e. and every $\beta > 0$.

II. Applications to semi-Dirichlet forms

- Assume that the semigroup $(P_t)_{t\geqslant 0}$ is associated to a semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$, where m is a σ-finite measure on the Lusin measurable space $(E, \mathcal{B})$.

- By [L. Beznea, N. Boboc, M. Röckner, *Pot. Anal.* 2006] there exists a (larger) Lusin topological space E_1 such that $E \subset E_1$, E belongs to $\mathcal{B}_1$ (the σ-algebra of all Borel subsets of E_1), $\mathcal{B} = \mathcal{B}_1|_E$, and $(\mathcal{E}, \mathcal{F})$ regarded as a semi-Dirichlet form on $L^2(E_1, \overline{m})$ is quasi-regular, where $\overline{m}$ is the measure on $(E_1, \mathcal{B}_1)$ extending m by zero on $E_1 \setminus E$. Consequently, we may consider a right Markov process X with state space E_1 which is associated with the semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$.

- If $u \in \mathcal{F}$ then $\widetilde{u}$ denotes a quasi continuous version of u as a function on E_1 which always exists and it is uniquely determined quasi everywhere.

For a closed set F define
$\mathcal{F}_{b,F} := \{v \in \mathcal{F} : v \text{ is bounded and } v = 0\ m\text{-a.e. on } E \setminus F\}$.

Theorem

Let $u \in \mathcal{F}$ and assume there exist a nest $(F_n)_{n\geq 1}$ and constants $(c_n)_{n\geqslant 1}$ such that

$$\mathcal{E}(u, v) \leqslant c_n \|v\|_\infty \quad \textit{for all } v \in \mathcal{F}_{b,F_n}.$$

Then $\widetilde{u}(X)$ is a $\mathbb{P}^x$-semimartingale for $x \in E_1$ quasi everywhere.

- If E is a bounded domain in $\mathbb{R}^d$ (or more generally in an abstract Wiener space) and the condition from the theorem holds for u replaced by the canonical projections, then the conclusion is that the underlying Markov process is a semimartingale.

- In particular, the semimartingale nature of reflected diffusions on general bounded domains can be studied.

This problem dates back to the work of
[R.F Bass, P. Hsu, *Proc. Amer. Math. Soc.* 1990]
where the authors showed that the reflected Brownian motion on a Lipschitz domain in $\mathbb{R}^d$ is a semimartingale.

- Later on, this result has been extended to more general domains and diffusions:
[R.J. Williams, W.A. Zheng, *Ann. Inst. Henri Poincaré* 1990],
[Z. Q. Chen, *Probab. Theory Related Fields*, 1993],
[Z.Q. Chen, PJ. Fitzsimmons, R.J. Williams, *Pot. Anal.* 1993], and
[E. Pardoux, R. J. Williams, *Ann. Inst. H. Poincaré Probab. Statist.* 1994]

- A clarifying result has been obtained in
[Z.Q. Chen, PJ. Fitzsimmons, R.J. Williams, *Pot. Anal.* 1993],
showing that the stationary reflecting Brownian motion on a bounded Euclidian domain is a quasimartingale on each compact time interval if and only if the domain is a strong Caccioppoli set.

- A complete study of these problems, but only in the symmetric case, have been done in a series of papers by M. Fukushima and co-authors, with deep applications to BV functions in both finite and infinite dimensions:
[M. Fukushima, *Electronic J. of Probability* 1999, *J. Funct. Anal.* 2000]
and
[M. Fukushima, M. Hino, *J. Funct. Anal.* 2001].

- All these previous results have been obtained using the same common tools: symmetric Dirichlet forms and Fukushima decomposition.

- Further applications to the reflection problem in infinite dimensions have been studied in
[M. Röckner, R. Zhu, X. Zhu, *Anna. Probab.* 2012] and
[M. Röckner, R. Zhu, X. Zhu, *Forum Math. 2015*]
where non-symmetric situations were also considered.

- In the case of semi-Dirichlet forms, a Fukushima decomposition is not yet known to hold, unless some additional hypotheses are assumed; see e.g. [Y. Oshima, Walter de Gruyter 2013].
Here is where our study played its role, allowing us to completely avoid Fukushima decomposition or the existence of the dual process.

The case of the local semi-Dirichlet forms

Assume that $(\mathcal{E}, \mathcal{F})$ is quasi-regular and that it is **local**, i.e., $\mathcal{E}(u, v) = 0$ for all $u, v \in \mathcal{F}$ with disjoint compact supports. The local property is equivalent with the fact that the associated process is a diffusion.

As in [M. Fukushima, *J. Funct. Anal.* 2000] the local property of $\mathcal{E}$ allows us to extend the results to the case when u is only locally in the domain of the form, or to even more general situation, as stated in the next result.

Corollary

Assume that $(\mathcal{E}, \mathcal{F})$ is local. Let u be a real-valued $\mathcal{B}$-measurable finely continuous function and let $(v_k)_k \subset \mathcal{F}$ such that $v_k \xrightarrow[k\to\infty]{} u$ pointwise except an m-polar set and boundedly on each element of a nest $(F_n)_{n\geqslant 1}$. Further, suppose that there exist constants c_n such that

$$|\mathcal{E}(v_k, v)| \leqslant c_n \|v\|_\infty \quad \text{for all } v \in \mathcal{F}_{b,F_n}.$$

Then $u(X)$ is a $\mathbb{P}^x$-semimartingale for $x \in E$ quasi everywhere.

III. Martingale functions with respect to the dual Markov process

Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of a right process X with state space E and let $\mathcal{T}_0$ be the Lusin topology of E, having $\mathcal{B}$ as Borel σ-algebra, and let m be a fixed $\mathcal{U}$- sub-invariant measure, i.e. $m \circ \alpha U_\alpha \leqslant m$, $\alpha > 0$.

Aim: To identify martingale functions and co-martingale ones, i.e., martingales w.r.t. some dual process.

- There exists a second sub-Markovian resolvent of kernels on E denoted by $\widehat{\mathcal{U}} = (\widehat{U}_\alpha)_{\alpha>0}$ which is in **weak duality** with $\mathcal{U}$ w.r.t. m in the sense that $\int_E fU_\alpha g\, dm = \int_E g\widehat{U}_\alpha f\, dm$ for all $f, g \geqslant 0$, and $\alpha > 0$.

- Both resolvents $\mathcal{U}$ and $\widehat{\mathcal{U}}$ can be contractively extended to any $L^p(E, m)$ space for all $1 \leqslant p < \infty$, and they are strongly continuous.

- There exist a larger Lusin measurable space $(\overline{E}, \overline{\mathcal{B}})$, with $E \subset \overline{E}$, $E \in \mathcal{B}$, $\mathcal{B} = \overline{\mathcal{B}}|_E$, and two processes $\overline{X}$ and $\widehat{X}$ with common state space $\overline{E}$, such that $\overline{X}$ is a right process with $\overline{E}$ endowed with a convenient Lusin topology having $\overline{\mathcal{B}}$ as Borel σ-algebra (resp. $\widehat{X}$ is a right process w.r.t. to a second Lusin topology on $\overline{E}$, also generating $\overline{\mathcal{B}}$), the restriction of $\overline{X}$ to E is precisely X, and the resolvents of $\overline{X}$ and $\widehat{X}$ are in duality with respect to $\overline{m}$, where $\overline{m}$ is the extension of m from E to $\overline{E}$ with zero on $\overline{E} \setminus E$.

- The α-excessive functions, $\alpha > 0$, with respect to $\widehat{X}$ on $\overline{E}$ are precisely the unique extensions by continuity in the fine topology generated by $\widehat{X}$ of the $\widehat{\mathcal{U}}_\alpha$-excessive functions.
In particular, the set E is dense in $\overline{E}$ in the fine topology of $\widehat{X}$.

- The strongly continuous resolvent of sub-Markovian contractions induced on $L^p(m)$, $1 \leqslant p < \infty$, by the process $\overline{X}$ (resp. $\widehat{X}$) coincides with $\mathcal{U}$ (resp. $\widehat{\mathcal{U}}$).

[L. Beznea, M. Röckner, *Pot. Anal.* 2015]
[L. Beznea, N. Boboc, M. Röckner, *Pot. Anal.* 2006]

Theorem

Let u be function from $L^p(E, m)$, $1 \leqslant p < \infty$. *Then the following assertions are equivalent.*

(i) The process $(u(X_t))_{t\geqslant 0}$ *is a martingale w.r.t.* $\mathbb{P}^x$ *for all* $x \in E$ *m-a.e.*

(ii) The process $(u(\widehat{X}_t))_{t\geqslant 0}$ *is a martingale w.r.t.* $\widehat{\mathbb{P}}^x$ *for all* $x \in E$ *m-a.e.*

(iii) The function u is L_p*-harmonic, i.e.* $u \in D(\mathsf{L}_p)$ *and* $\mathsf{L}_p u = 0$.

(iv) The function u is $\widehat{\mathsf{L}}_p$*-harmonic, i.e.* $u \in D(\widehat{\mathsf{L}}_p)$ *and* $\widehat{\mathsf{L}}_p u = 0$.

IV. Excessive and invariant functions on L^p-spaces

Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on E and m is a σ-finite sub-invariant measure. Let $\widehat{\mathcal{U}} = (\widehat{U}_\alpha)_{\alpha>0}$ be a second sub-Markovian resolvent of kernels on E which is in weak duality with $\mathcal{U}$ w.r.t. m.

We focus on a special class of differences of excessive functions (which are in fact harmonic when the resolvent is Markovian).

- A real-valued $\mathcal{B}$-measurable function $v \in \bigcup_{1\leqslant p\leqslant\infty} L^p(E,m)$ is called **$\mathcal{U}$-invariant** provided that $U_\alpha(vf) = vU_\alpha f$ m-a.e. for all bounded and $\mathcal{B}$-measurable functions f and $\alpha > 0$.

- A set $A \in \mathcal{B}$ is called **$\mathcal{U}$-invariant** if 1_A is $\mathcal{U}$-invariant; the collection of all $\mathcal{U}$-invariant sets is a σ-algebra.

• If $v \geqslant 0$ is $\mathcal{U}$-invariant then there exists $u \in E(\mathcal{U})$ such that $u = v$ m-a.e.

• If $\alpha U_\alpha 1 = 1$ m-a.e. then for every invariant function v we have $\alpha U_\alpha v = v$ m-a.e, which is equivalent (if $\mathcal{U}$ is strongly continuous) with v being L_p-harmonic, i.e. $v \in D(\mathsf{L}_p)$ and $\mathsf{L}_p v = 0$.

The next result is a straightforward consequence of the duality between $\mathcal{U}$ and $\widehat{\mathcal{U}}$.

Proposition

The following assertions hold.
(i) A function u is $\mathcal{U}$-invariant if and only if it is $\widehat{\mathcal{U}}$-invariant.

(ii) The set of all $\mathcal{U}$-invariant functions from $L^p(E, m)$ is a vector lattice with respect to the pointwise order relation.

Theorem

Let $u \in L^p(E, m)$, $1 \leqslant p < \infty$, *and consider the following conditions.*

(i) $\alpha U_\alpha u = u$ *m-a.e. for one (and thus for all)* $\alpha > 0$.

(ii) $\alpha \widehat{U}_\alpha u = u$ *m-a.e.,* $\alpha > 0$.

(iii) The function u is $\mathcal{U}$*-invariant.*

(iv) $U_\alpha u = uU_\alpha 1$ *and* $\widehat{U}_\alpha u = u\widehat{U}_\alpha 1$ *m-a.e. for one (and thus for all)* $\alpha > 0$.

(v) The function u is measurable w.r.t. the σ*-algebra of* $\mathcal{U}$*-invariant sets.*

Then $\mathcal{I}_p := \{u \in L^p(E, m) : \alpha U_\alpha u = u \text{ m-a.e.}, \ \alpha > 0\}$ *is a vector lattice w.r.t. the pointwise order relation and (i)* $\Leftrightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Leftrightarrow$ *(iv)* $\Leftrightarrow$ *(v).*
If $\alpha U_\alpha 1 = 1$ *or* $\alpha \widehat{U}_\alpha 1 = 1$ *m-a.e. then assertions (i) - (v) are equivalent.*
If $m(E) < \infty$ *and* $p = \infty$ *then all of the statements above are still true.*
If $p = \infty$ *and* $\mathcal{U}$ *is m-recurrent (i.e. there exists* $0 \leqslant f \in L^1(E, m)$ *s.t.* $Uf = \infty$ *m-a.e.) then the equivalences of (i)-(v) remain valid.*

- Similar characterizations for invariance as in the above theorem, but in the recurrent case and for functions which are bounded or integrable with bounded negative parts were investigated in
[R. L. Schilling, *Probab. Math. Statist.*, 2004].

- Of special interest is the situation when the only invariant functions are the constant ones (*irreducibility*) because it entails ergodic properties for the semigroup resp. resolvent; see e.g.
[K.T. Sturm, *J. Reine Angew. Math.* 1994],
[S. Albeverio, Y. G. Kondratiev, and M. Röckner, *J. Funct. Anal.* 1997], and
[L. Beznea, I. Cîmpean, M. Röckner, *Irreducible recurrence, ergodicity, and extremality of invariant measures for resolvents*, arXiv:1409.6492v2].