Heat kernels for non-symmetric diffusions operators with jumps

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Collaborators

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Motivation

Given a Markov process $(X, \mathbb{P}_x, x \in E)$ on E, its transition semigroup P_t is given by

$$P_t f(x) = \mathbb{E}_x[f(X_t)].$$

The infinitesimal generator $\mathcal{L}$ of X is

$$\mathcal{L} f(x) = \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t}.$$

Suppose X has a transition density function $p(t, x, y)$ with respect to a measure m on E; that is,

$$\mathbb{E}_x[f(X_t)] = \int_E p(t, x, y) f(y) m(dy).$$

The kernel $p(t, x, y)$ is the fundamental solution, also called heat kernel, of $\mathcal{L}$.

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Heat kernel

Transition density function $p(t, x, y)$ of X encodes all the information of the process. However it is impossible to get its explicit exact formula except for some very few special cases such as Brownian motion, OU processes, and Cauchy processes. Thus obtaining its two-sided sharp estimates is an important research subject both in probability theory and in analysis.

HK for Elliptic Operators

Heat kernels for second order elliptic operators have been well studied and there are many beautiful results. For example, when

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

where $(a_{ij}(x))_{1 \leq i,j \leq n}$ is a measurable $d \times d$ matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic and bounded, there is a symmetric diffusion X on $\mathbb{R}^d$ having $\mathcal{L}$ as its L^2-infinitesimal generator. The diffusion process X has a transition density function $p(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$:

$$\mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy.$$

DeGiorgi-Nash-Moser-Aronson theory

Every bounded parabolic function of $\mathcal{L}$ (or equivalently, of X) is locally Hölder continuous and the parabolic Harnack inequality holds for non-negative parabolic functions of $\mathcal{L}$. Moreover, $\mathcal{L}$ has a jointly continuous heat kernel $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ that enjoys the following Aronson's estimate: there are constants $c_k > 0$, $k = 1, \cdots, 4$, so that

$$c_1 \, p^c(t, c_2|x - y|) \le p(t, x, y) \le c_3 \, p^c(t, c_4|x - y|)$$

for every $t > 0$ and $x, y \in \mathbb{R}^d$. Here

$$p^c(t, r) := t^{-d/2} \exp(-r^2/t).$$

Goal of This Talk

is to present some recent development of the DeGiorgi-Nash-Moser-Aronson type theory for the following time-inhomogeneous and non-symmetric non-local operator:

$$\mathcal{L}_t f(x) := \mathcal{L}_t^a f(x) + b_t \cdot \nabla f(x) + \mathcal{L}_t^\kappa f(x), \qquad (1)$$

where

$$\mathcal{L}_t^a f(x) := \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t,x) \partial_{ij}^2 f(x), \; b_t \cdot \nabla f(x) := \sum_{i=1}^d b_i(t,x) \partial_i f(x),$$

$$\mathcal{L}_t^\kappa f(x) := \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - 1_{\{|z| \le 1\}} z \cdot \nabla f(x) \right) \frac{\kappa(t,x,z)}{|z|^{d+\alpha}} \mathrm{d}z.$$

Here $a(t,x) := (a_{ij}(t,x))_{1 \le i,j \le d}$ is a $d \times d$-symmetric matrix-valued measurable function on $[0,\infty) \times \mathbb{R}^d$, $b(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\kappa(t,x,z) : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are measurable functions, and $\alpha \in (0,2)$.

Why study non-local operator?

From analytic side:

The well-known Courrège theorem states that a linear operator $\mathcal{L}$ on $C_\infty(\mathbb{R}^d)$ with domain $\mathrm{Dom}(\mathcal{L}) \supset C_c^\infty(\mathbb{R}^d)$ satisfies the positive maximum principle if and only if $\mathcal{L}$ takes the following form

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^2 f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x) + c(x)f(x)$$
$$+ \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - 1_{\{|z|\le 1\}} z\cdot\nabla f(x)\right)\mu_x(\mathrm{d}z),$$

where $a = (a_{ij}(x))_{1\le i,j\le d}$ is a $d \times d$-symmetric positive definite matrix-valued measurable function on $\mathbb{R}^d$, $b(x): \mathbb{R}^d \to \mathbb{R}^d$, $c: \mathbb{R}^d \to (-\infty, 0]$ are measurable functions and $\mu_x(\mathrm{d}z)$ is a family of Lévy measures, with that a, b, c, μ enjoy some continuity with respect to x

Why study non-local operator?

From Probabilistic side: Discontinuous Markov processes such as Lévy processes.

Probabilistic meaning of $j(x, z) = \frac{\kappa(t,x,z)}{|z|^{d+\alpha}}$: jumping intensity from x to $x + z$.

From Applications: Anomalous superdiffusion when the "random walker" remains in motion without changing direction for a time that follows a distribution that has heavy tail (e.g. Pareto-Lévy distribution).

Bird search: more effective

UCLA burglary hotspot model: study and predict burglary location

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Symmetric non-local operator

Symmetric operator on $\mathbb{R}^d$:

$$\mathcal{L}f(x) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \mathcal{L}^{(j)}f(x),$$

where $a_{ij}(x) = a_{ji}(x)$ and

$$\mathcal{L}^{(j)}f(x) = \lim_{\varepsilon\to 0}\int_{\{y\in\mathbb{R}^d : |y-x|>\varepsilon\}} (f(y)-f(x))J(x,y)\mathrm{d}y.$$

Here $J(x,y) \geq 0$ is a symmetric kernel, which is the jumping intensity from x to y.

Some history: symmetric case

- When $a = 0$ and $J(x, y) \asymp \frac{1}{|x-y|^{d+\alpha}}$ for $0 < \alpha < 2$, [Chen and Kumagai (2003)] showed:

$$p(t, x, y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$.

- When $a = 0$ and $J(x, y) \asymp \frac{1}{|x-y|^d \phi(|x-y|)}$ where ϕ is a strictly increasing function on $\mathbb{R}_+$ with $\phi(0) = 0$ and $\phi(1) = 1$ so that

$$c_1 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq c_2 \left(\frac{R}{r}\right)^{\beta_2} \quad \text{for } 0 < r < R < \infty,$$

for $0 < \beta_1 \leq \beta_2 < 2$, [Chen and Kumagai (2008)] showed:

$$p(t, x, y) \asymp \frac{1}{\phi^{-1}(t)^d} \wedge \frac{t}{|x - y|^d \phi(|x - y|)} =: p^j(t, |x - y|)$$

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\begin{aligned}
& c_1 \left(t^{-d/2} \wedge \phi^{-1}(t)^{-d} \right) \wedge \left(p^c(t, c_2|x-y|) + p^j(t, |x-y|) \right) \\
\le \; & p(t,x,y) \\
\le \; & c_3 \left(t^{-d/2} \wedge \phi^{-1}(t)^{-d} \right) \wedge \left(p^c(t, c_4|x-y|) + p^j(t, |x-y|) \right).
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• Approach: Dirichlet form, Nash's inequality, Davies method, probabilistic approach by estimating various hitting probabilities.

• These are stability results.

• Can be viewed as DeGiorgi-Nash-Moser-Aronson type theory for symmetric non-local operators.

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- Chen and Zhang [PTRF, 2016] obtained two-sided HKE for

$$\mathcal{L}^\kappa f(x) := p.v. \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \nabla f(x) \cdot z 1_{\{|z| \le 1\}} \right) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \mathrm{d}z$$

  on any fixed time intervals, where $\kappa(x, z) = \kappa(x, -z)$, $|\kappa(x, z) - \kappa(y, z)| \le C|x - y|^\beta$ for some $\beta \in (0, 1)$.
- Recently, Kim, Song and Vondracek use the approach of Chen and Zhang (2016) to obtain HKE for

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$$\int_{|z|\le r} z\kappa(t,x,z)\mathrm{d}z = 0 \quad \text{for any } r>0.$$

when $\alpha = 1$

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Goal

Our main aim is to establish two-sided estimates for general operator $\mathcal{L}_t f(x) := \mathcal{L}_t^a f(x) + b_t \cdot \nabla f(x) + \mathcal{L}_t^\kappa f(x)$ under some mild assumptions on a, b and κ.

Such operators have also been studied extensively in analysis, e.g., by Caffarelli, Silvestre, etc.

While heat kernels of differential operators have been studied extensively and there are quite many progress recently for symmetric non-local operators, there are very limited results on heat kernels for non-symmetric non-local operators.

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Assumptions

Below we assume $d \geq 2$ and make the following assumptions on a and κ:

(H^a) There are $c_1 > 0$ and $\beta \in (0, 1)$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d$,

$$|a(t, y) - a(t, x)| \leq c_1 |y - x|^\beta, \tag{2}$$

and for some $c_2 \geq 1$,

$$c_2^{-1} \mathbb{I}_{d\times d} \leq a(t, x) \leq c_2 \mathbb{I}_{d\times d}. \tag{3}$$

Here $\mathbb{I}_{d\times d}$ denotes the $d \times d$ identity matrix.

(H$^\kappa$) $\kappa(t, x, z)$ is a bounded measurable function and if $\alpha = 1$, we require for any $r > 0$,

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It is easy to see that for each fixed $T > 0$, the two-sided estimates for symmetric jumps with stable-like jumps ($J(x, y) \asymp 1/|x-y|^{d+\alpha}$) on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ is equivalent to

$$\begin{aligned} & \widetilde{c}_1 \left(t^{-d/2} e^{-c_2|x-y|^2/t} + \frac{t}{(t^{1/2}+|x-y|)^{d+\alpha}} \right) \\ \leq \quad & p(t,x,y) \leq \widetilde{c}_3 \left(t^{-d/2} e^{-c_4|x-y|^2/t} + \frac{t}{(t^{1/2}+|x-y|)^{d+\alpha}} \right). \end{aligned}$$

Let

$$\xi_{\lambda,\gamma}(t,x) := t^{(\gamma-d)/2} e^{-\lambda|x|^2/t}, \quad \eta_{\alpha,\gamma}(t,x) := \frac{t^{\gamma/2}}{(t^{1/2}+|x|)^{d+\alpha}}.$$

Then for $0 < t \leq T$ and $x, y \in \mathbb{R}^d$,

$$\widetilde{c}_1 \left(\xi_{c_2,0}(t,x) + \eta_{\alpha,2}(t,x) \right) \leq p(t,x,y) \leq \widetilde{c}_3 \left(\xi_{c_4,0}(t,x) + \eta_{\alpha,2}(t,x) \right).$$

Kato class

Definition 1 (Generalized Kato's class)

For $\alpha \in [1, +\infty)$, define

$$\mathbb{K}_\alpha := \left\{ f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \lim_{\delta \to 0} K^f_\alpha(\delta) = 0 \right\},$$

$$\overline{\mathbb{K}}_\alpha := \left\{ f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \overline{K}^f_\alpha(1) < \infty \right\},$$

where

$$K^f_\alpha(\delta) := \sup_{(t,x)} \int_0^\delta \int_{\mathbb{R}^d} |f|(t \pm s, x \pm y)\eta_{\alpha,\alpha-1}(s,y)\mathrm{d}y\mathrm{d}s,$$

$$\overline{K}^f_\alpha(\delta) := \sup_{(t,x)} \int_0^\delta \int_{\mathbb{R}^d} |f|(t \pm s, x \pm y)\eta_{\alpha,0}(s,y)\mathrm{d}y\mathrm{d}s.$$

If $\beta > \alpha \geq 1$, then $\eta_{\beta,\beta-1}(t,x) \leq \eta_{\alpha,\alpha-1}(t,x)$ and so $\mathbb{K}_\alpha \subset \mathbb{K}_\beta$.

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$$\mathbb{K}_\alpha := \left\{ f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \lim_{\delta \to 0} K^f_\alpha(\delta) = 0 \right\},$$
$$\overline{\mathbb{K}}_\alpha := \left\{ f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \overline{K}^f_\alpha(1) < \infty \right\},$$

where

$$K^f_\alpha(\delta) := \sup_{(t,x)} \int_0^\delta \int_{\mathbb{R}^d} |f|(t \pm s, x \pm y)\eta_{\alpha,\alpha-1}(s,y)\mathrm{d}y\mathrm{d}s,$$
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If $\beta > \alpha \geq 1$, then $\eta_{\beta,\beta-1}(t,x) \leq \eta_{\alpha,\alpha-1}(t,x)$ and so $\mathbb{K}_\alpha \subset \mathbb{K}_\beta$.

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Kato class

Definition 1 (Q.S. Zhang [MM, 1997])

For $\alpha = +\infty$, we define

$$\mathbb{K}_\infty := \left\{ f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R} \text{ satisfies } \lim_{\delta\to 0} N^f_\lambda(\delta) = 0 \ \text{ for every } \lambda > 0 \right\},$$

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Remark: If $\frac{d}{p} + \frac{2}{q} < 1$, then

$$L^q(\mathbb{R}; L^p(\mathbb{R}^d)) \subset \mathbb{K}_1 \subset \mathbb{K}_\alpha \subset \mathbb{K}_\infty;$$

and if $\alpha \in (1, 2)$ and $\frac{d}{p} + \frac{2}{q} < 2 - \alpha$, then

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Duhamel's formula

- Let $Z(t, x; s, y)$ be the fundamental solution of $\{\mathcal{L}_t^a; t \geq 0\}$.
- Let $\mathcal{L}_t^{b,\kappa} := b \cdot \nabla + \mathcal{L}_t^\kappa$. Since $\mathcal{L}_t$ can be viewed as a perturbation of $\mathcal{L}_t^a$ by $\mathcal{L}_t^{b,\kappa}$, heuristically the fundamental solution $p(t, x; s, y)$ of $\mathcal{L}_t$ should satisfy the following Duhamel's formula: for all $0 \leq t < s < \infty$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} & p(t, x; s, y) \\ = \; & Z(t, x; s, y) + \int_t^s \int_{\mathbb{R}^d} p(t, x; r, z) \mathcal{L}_r^{b,\kappa} Z(r, \cdot; s, y)(z) \mathrm{d}z \mathrm{d}r, \end{aligned} \tag{5}$$

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Main results

For any $T \in (0, \infty]$ and $\varepsilon \in [0, T)$, we write

$$\mathbb{D}_\varepsilon^T := \Big\{(t, x; s, y) : x, y \in \mathbb{R}^d \text{ and } s, t \geq 0 \text{ with } \varepsilon < s - t < T\Big\}.$$

Theorem 1

Let $\alpha \in (0, 2)$. Under **($\mathbf{H}^a$)**, **($\mathbf{H}^\kappa$)** and $b \in \mathbb{K}_2$, there is a unique continuous function $p(t, x; s, y)$ on $\mathbb{D}_0^\infty$ that satisfies (5), and

- (Upper-bound estimate) For any $T > 0$, there exist constants $C_0, \lambda_0 > 0$ such that on $\mathbb{D}_0^T$,

$$|p(t, x; s, y)| \leq C_0(\xi_{\lambda_0,0} + \|\kappa\|_\infty \eta_{\alpha,2})(s - t, y - x).$$

- (C-K equation) For all $0 \leq t < r < s < \infty$ and $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p(t, x; r, z) p(r, z; s, y) \mathrm{d}z = p(t, x; s, y).$$

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Main results

Theorem 1 (continuing)

- (Gradient estimate) For any $T > 0$, there exist constants $C_1, \lambda_1 > 0$ such that on $\mathbb{D}_0^T$,

$$|\nabla_x p(t,x;s,y)| \le C_1(\xi_{\lambda_1,-1} + \|\kappa\|_\infty \eta_{\alpha,1})(s-t, y-x).$$

- (Fractional derivative estimate) If in addition for $\alpha \in (0,1]$, $b \in \mathbb{K}_1$ and for $\alpha \in (1,2)$, $b \in \overline{\mathbb{K}}_\alpha$, then for any $T > 0$, there exists a constant $C_2 > 0$ such that on $\mathbb{D}_0^T$,

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Main results

Theorem 1 (continuing)

- (Conservativeness) For any $0 \leq t < s < \infty$ and $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p(t, x; s, y)\mathrm{d}y = 1.$$

- (Generator) For any $f \in C_b^2(\mathbb{R}^d)$, we have

$$P_{t,s}f(x) - f(x) = \int_t^s P_{t,r}\mathcal{L}_r f(x)\mathrm{d}r,$$

  where $P_{t,s}f(x) := \int_{\mathbb{R}^d} p(t, x; s, y) f(y)\mathrm{d}y$.
- (Continuity) For any bounded and uniformly continuous function $f(x)$,

$$\lim_{|t-s|\to 0} \|P_{t,s}f - f\|_\infty = 0.$$

Theorem 2

Under the same assumptions of Theorem 1, if for each $t > 0$ and $x \in \mathbb{R}^d$,

$$\kappa(t, x, z) \geq 0, \quad a.e.\ z \in \mathbb{R}^d, \tag{6}$$

then $p(t, x; s, y) \geq 0$ on $\mathbb{D}_0^\infty$. Moreover, for any $T > 0$, there are constants $C_3, \lambda_3 > 0$ such that

$$p(t, x; s, y) \geq C_3(\xi_{\lambda_3,0} + m_\kappa \eta_{\alpha,2})(s - t, y - x) \text{ on } \mathbb{D}_0^T,$$

where $m_\kappa := \inf_{(t,x)} \operatorname{essinf}_{z \in \mathbb{R}^d} \kappa(t, x, z)$. If in addition, κ satisfies that for each $t \geq 0$,

$$x \mapsto \kappa(t, x, z) \text{ is continuous} \quad a.e.\ z \in \mathbb{R}^d,$$

then (6) is also a necessary condition to the positivity of $p(t, x; s, y)$.

Lower bound estimates

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Finite range juming kernel case

Theorem 3

Suppose that **(H^a)**, **(H$^\kappa$)** and $b \in \mathbb{K}_2$ hold. Let $T > 0$, and for $\lambda > 0$, set

$$\overline{\eta}_{\alpha,\lambda}(t,x) := \frac{t}{(t^{1/2} + |x|)^{d+\alpha}} 1_{\{|x| \le 1/2\}} + (t/|x|)^{\lambda |x|} 1_{\{|x|>1/2\}}.$$

(i) If in addition $0 \le \kappa(t,x,z) \le \kappa_0 1_{|z|\le 1}(z)$ for some $\kappa_0 > 0$, then there are constants $C_1, \lambda_1 > 0$ such that

$$p(t,x;s,y) \le C_1 \left(\xi_{\lambda_1,0} + \overline{\eta}_{\alpha,1/8} \right)(s-t, y-x) \quad \text{on } \mathbb{D}_0^T.$$

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Sketch of Approach

- To solve the integral equation (5), let $p_0(t,x;s,y) := Z(t,x;s,y)$, and for $n \in$, define

$$p_n(t,x;s,y) := \int_t^s \int_{\mathbb{R}^d} p_{n-1}(t,x;r,z)\mathcal{L}_r^{b,\kappa} Z(r,\cdot;s,y)(z)\mathrm{d}z\mathrm{d}r. \tag{7}$$

- Formally, we have

$$p(t,x;s,y) = \sum_{n=0}^{\infty} p_n(t,x;s,y).$$

- Key point is to estimate the fractional derivative $\mathcal{L}^\kappa Z(r,\cdot;s,y)(z)$ and then the convotion type estimate in (7). Hard analysis.

Application to SDE driven by Lévy processes, which are typically non-symmetric.

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