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On comparability of nonlocal energy forms

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Story I

α-stable processes in an anisotropic medium

Imagine a Markov process that, at every point $x \in \mathbb{R}^d$, jumps like an α-stable process but only within an double-cone with apex at x. The double-cones are given and may depend on x as they wish.

Questions:

- How to construct such a process?
- Properties of the process, e.g., Feller property?
- Differences resp. similarities with rotationally symmetric α-stable process?
- Motivation and applications?

Story II

Quadratic forms on L^2

Assume $a_{ij} : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\Lambda^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad (x, \xi \in \mathbb{R}^d)$$

for some $\Lambda > 1$.

Then

$$\int a_{ij}\, \partial_i f\, \partial_j f \asymp \int |\nabla f|^2 = [f]_{H^1}$$

for every $f \in L^2(\mathbb{R}^d)$.

Question:

How does an analogous observation look like for fractional Sobolev spaces $H^{\alpha/2}(\mathbb{R}^d), 0 < \alpha < 2$?

Simple case: Assume $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\Lambda^{-1}|x-y|^{-d-\alpha} \leq k(x,y) \leq \Lambda|x-y|^{-d-\alpha} \qquad (x, y \in \mathbb{R}^d, x \neq y) \tag{1}$$

for some $\Lambda > 1$ and $\alpha \in (0,2)$. Then, obviously,

$$\iint (f(x)-f(y))^2 \underbrace{\mathbf{k(x,y)}\,\mathrm{d}\mathbf{y}}_{\mu(\mathbf{x},\mathrm{d}\mathbf{y})} \mathrm{d}x \asymp \iint \frac{(f(x)-f(y))^2}{|\mathbf{x}-\mathbf{y}|^{\mathbf{d}+\alpha}} \,\mathrm{d}\mathbf{y}\,\mathrm{d}x = [f]_{H^{\alpha/2}} \tag{2}$$

for $f \in L^2(\mathbb{R}^d)$.

Simple case: Assume $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\Lambda^{-1}|x-y|^{-d-\alpha} \le k(x,y) \le \Lambda|x-y|^{-d-\alpha} \qquad (x, y \in \mathbb{R}^d, x \neq y) \tag{1}$$

for some $\Lambda > 1$ and $\alpha \in (0,2)$. Then, obviously,

$$\iint (f(x)-f(y))^2 \underbrace{\mathbf{k(x,y)\,dy}}_{\mu(\mathbf{x},\mathrm{d}\mathbf{y})} \mathrm{d}x \asymp \iint \frac{(f(x)-f(y))^2}{|\mathbf{x}-\mathbf{y}|^{\mathbf{d}+\alpha}} \,\mathrm{d}\mathbf{y}\,\mathrm{d}x = [f]_{H^{\alpha/2}} \tag{2}$$

for $f \in L^2(\mathbb{R}^d)$.

Main question

Under which conditions, more general than (1), does (2) hold?

Remark:

- The question can be posed in the more general context where $k(x,y)\,\mathrm{d}y$ is replaced by a family of measures $\mu(x, \mathrm{d}y)$.
- We derive a concrete case of interest by looking at the **Boltzmann equation**.

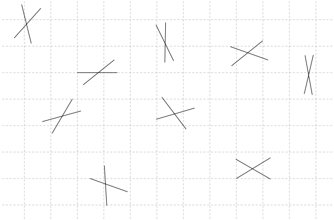
Setup.

- By V we denote a double cone in $\mathbb{R}^d$ with apex at $0 \in \mathbb{R}^d$, symmetry axis $v \in \mathbb{R}^d$ and apex angle $\vartheta \in (0, \frac{\pi}{2}]$.
- Let $\mathcal{V} = (0, \frac{\pi}{2}] \times \mathbb{P}^{d-1}_{\mathbb{R}}$ denote the family of all such double cones.
- For $x \in \mathbb{R}^d$ we define a shifted double cone by $V[x] = V + x$.
- A mapping $\Gamma : \mathbb{R}^d \to \mathcal{V}$ is called a configuration. If the infimum ϑ over all apex angles of cones in $\Gamma(\mathbb{R}^d)$ is positive, then Γ is called ϑ-bounded.
- Γ is called ϑ-admissible if it is ϑ-bounded and

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mid y - x \in \Gamma(x)\} \text{ is a Borel set in } \mathbb{R}^d \times \mathbb{R}^d\,. \qquad \text{(A)}$$

- For $x \in \mathbb{R}^d$ and Γ a configuration, we define $V^\Gamma[x] = x + \Gamma(x)$.

$V^\Gamma[x]$ for different x and one fixed configuration.



Short discussion of condition (A)

Γ is called ϑ-admissible if it is ϑ-bounded and

$$\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid y - x \in \Gamma(x)\} \text{ is a Borel set in } \mathbb{R}^d \times \mathbb{R}^d . \qquad \text{(A)}$$

- $y - x \in \Gamma(x) \;\Leftrightarrow\; y \in V^\Gamma[x] \;\Leftrightarrow\;$ "y can be seen from x"
- (A) says that the function $(x,y) \mapsto \mathbb{1}_{V^\Gamma[x]}(y) + \mathbb{1}_{V^\Gamma[y]}(x)$ is Borel measurable.
- A non-obvious consequence of condition (A) is that for every Borel set $B \subset \mathbb{R}^d$ the set

  $$\{x \in \mathbb{R}^d | B \subset \Gamma(x)\}$$

  is Lebesgue-measurable [cf. Debreu 1967]. In particular, one can choose B as one of the elements in $\Gamma(\mathbb{R}^d)$ or any other double-cone $V \in \mathcal{V}$.

Theorem (Debreu, 1967): Assume X is a complete measure space and Y is a Polish space. Assume $F \subset X \times Y$ is measurable. Then, for every Borel set $B \subset Y$, the set $\{x \in X | \exists b \in B : (x,b) \in F\}$ is measurable.

Corollary: The set $\{x \in X | \forall b \in B : (x,b) \in F\}$ is measurable.

Theorem 1

Let Γ be a ϑ-admissible configuration. Let $k : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ be a measurable function satisfying $k(x, y) = k(y, x)$ and

$$\Lambda^{-1}\big(\mathbb{1}_{V^{\Gamma}[x]}(y) + \mathbb{1}_{V^{\Gamma}[y]}(x)\big)|x - y|^{-d-\alpha} \leq k(x, y) \leq \Lambda|x - y|^{-d-\alpha}, \qquad (3)$$

for all x and y, where $\Lambda \geq 1$ is some constant. Then there is a constant $c > 0$ such that for every ball $B \subset \mathbb{R}^d$ and for every $f \in L^2(B)$

$$c \int\limits_{B\times B} (f(x) - f(y))^2 |x - y|^{-d-\alpha} \mathrm{d}(x, y) \leq \int\limits_{B\times B} (f(x) - f(y))^2 k(x, y) \mathrm{d}(x, y)\,. \qquad (4)$$

The constant c depends on Λ, d, and on ϑ. It is independent of k and Γ.

Remark: [Imbert/Silvestre, 2016] provides a similar result, cf. [Silvestre, 2016] for the derivation of the setup from the Boltzmann equation.

Main difference: In our setup, two cones might not intersect. On the other hand, our cones are “nice”.

Corollary 2: *For every ball $B \subset \mathbb{R}^d$ and each $f \in L^2(B)$*

$$\iint_{B\times B} (f(x) - f(y))^2 k(x,y) \,\mathrm{d}y \,\mathrm{d}x \asymp [f]_{H^{\alpha/2}(B)}.$$

The constant depends only on Λ and ϑ. In particular, it is independent of B.

Corollary 3: *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$ with $\mathcal{F} = H^{\alpha/2}(\mathbb{R}^d)$ and*

$$\mathcal{E}(f,g) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} (f(y) - f(x)) (g(y) - g(x)) k(x,y) \,\mathrm{d}y \,\mathrm{d}x,$$

is a regular Dirichlet form. There exists a corresponding Hunt process.

Corollary 4 [Dyda/Kassmann]: *Weak solutions $u \in H^{\alpha/2}(\Omega)$, $0 < \alpha < 2$, to*

$$\mathcal{E}(u, \varphi) = (f, \varphi) \qquad (\varphi \in C_c^\infty(\Omega))$$

satisfy Hölder a-priori estimates in the interior of Ω.

Some parts of the proof.

Outline of the proof

1. Reduction to a finite family of cones
2. Derivation of a corresponding discrete result: Theorem 7
3. Finding chains that connect any two points in $\mathbb{Z}^d$
4. Renormalization
5. Proof of Theorem 7

1. Reduction to a finite family of cones

Lemma 5

There are numbers $L \in \mathbb{N}$ and $\theta \in (0, \frac{\pi}{2}]$, and double cones $V^1, ..., V^L$ centered at 0 with apex angle θ and symmetry axis $v^1, \dots, v^L \in \mathcal{S}^{d-1}$ such that

$$\forall x \in \mathbb{R}^d \; \exists m \in \{1, \dots, L\} : V^m \subset \Gamma(x).$$

The constants L and θ depend on the dimension d and ϑ but not on Γ itself.

Proof: Obviously

$$\mathcal{S}^{d-1} \subset \bigcup_{v \in \mathcal{S}^{d-1}} V\left(v, \frac{\vartheta}{3}\right),$$

Since the right hand side is an open cover of $\mathcal{S}^{d-1}$, we can choose $v^1, ..., v^L \in \mathcal{S}^{d-1}$ with

$$\mathcal{S}^{d-1} \subset \bigcup_{m=1}^{L} V\left(v^m, \frac{\vartheta}{3}\right).$$

Define *reference cones* $V^m = V\left(v^m, \frac{\vartheta}{3}\right)$ for $m = 1, ..., L$ and set $\theta = \vartheta/3$. □

2. Derivation of a corresponding discrete result: Theorem 7.

Let us derive a discrete $\mathbb{Z}^d$-version of Theorem 1. Set $\omega_h^k : h\mathbb{Z}^d \times h\mathbb{Z}^d \to \mathbb{R}$ by

$$\omega_h^k(x,y) = h^{-2d} \int\limits_{A_h(x)\times A_h(y)} k(s,t)\,\mathrm{d}(s,t)\,,$$

where $A_h(y)$ is a cube centered at y.

Proposition 6

There are constants $C = C(d,\vartheta) > 0$ and $\vartheta' \in (0, \frac{\pi}{2}]$ and a ϑ'-bounded configuration Γ' such that for all $x, y \in \mathbb{Z}^d$ with $|x-y| \geq \frac{1}{2}\sqrt{d}$

$$C\Lambda^{-1}\left(\mathbb{1}_{V^{\Gamma'}[x]}(y) + \mathbb{1}_{V^{\Gamma'}[y]}(x)\right)|x-y|^{-d-\alpha} \leq \omega_1^k(x,y)$$

The angle ϑ' depends only on θ and ϑ. There is no further dependence on Γ.

Strategy: Via scaling we obtain a discrete $h\mathbb{Z}^d$-version of Theorem 1. We will prove this result and obtain Theorem 1 after letting $h \to 0$.

Theorem 1

There is a constant $c > 0$ such that for every ball $B \subset \mathbb{R}^d$ and $f \in L^2(B)$

$$c \int_{B \times B} (f(x) - f(y))^2 |x - y|^{-d-\alpha} \mathrm{d}(x, y) \leq \int_{B \times B} (f(x) - f(y))^2 k(x, y) \mathrm{d}(x, y) \,. \qquad (5)$$

The constant c is independent of k and Γ.

The discrete $\mathbb{Z}^d$-version is as follows.

Theorem 7

Let $R_0 > 0$. For some $\kappa, c > 0$, every ball B and function $f : (\kappa B \cap \mathbb{Z}^d) \to \mathbb{R}$

$$c \sum_{\substack{x \in B \cap \mathbb{Z}^d,\, y \in B \cap \mathbb{Z}^d \\ |x-y| \geq R_0}} (f(x) - f(y))^2 |x - y|^{-d-\alpha} \leq \sum_{\substack{x \in \kappa B \cap \mathbb{Z}^d,\, y \in \kappa B \cap \mathbb{Z}^d \\ |x-y| \geq R_0}} (f(x) - f(y))^2 \omega(x, y).$$

Here, $\omega(x, y)$ is as in Proposition 6, e.g., $\omega(x, y) = \omega_1^k(x, y)$.

3. Finding chains that connect any two points in $\mathbb{Z}^d$.

Main tool here: Defining a graph structure

From a configuration $\Gamma : \mathbb{Z}^d \to \mathcal{V}$ we construct an undirected graph G as follows: the vertex set is $\mathbb{Z}^d$ and there is an edge from x to y if $y \in V^\Gamma[x]$ or $x \in V^\Gamma[y]$.

Question: Is G connected as an *undirected* graph if the underlying configuration is ϑ-bounded?

WLOG we assume that the image of Γ contains only a finite number of elements, i.e., finitely many types (directions) with only one apex angle. Thus, crucial parts of the argument can be proved by induction on the number of cones in $\Gamma(\mathbb{R}^d)$.

Proposition 8

Every two lattice points in a given ball of radius r are connected via an edge path that does not leave a larger ball of radius R. R depends only on r, ϑ and d.

4. Renormalization.

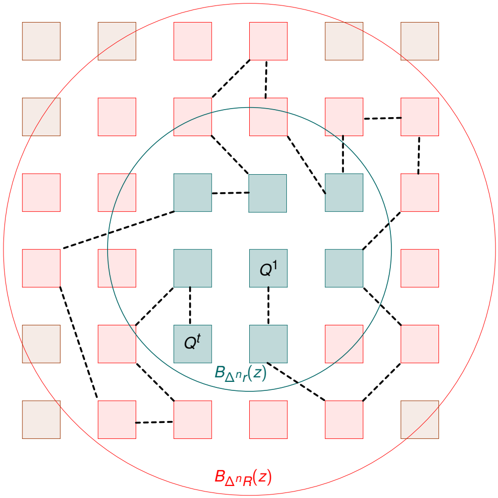
Theorem 9

Let $R_0 > 0$. There exist positive numbers N and M and a constant $\lambda \geq R_0$, all independent of Γ, and a collection $(p_{xy})_{x,y\in\mathbb{Z}^d}$ of unoriented edge paths in G such that the following holds:

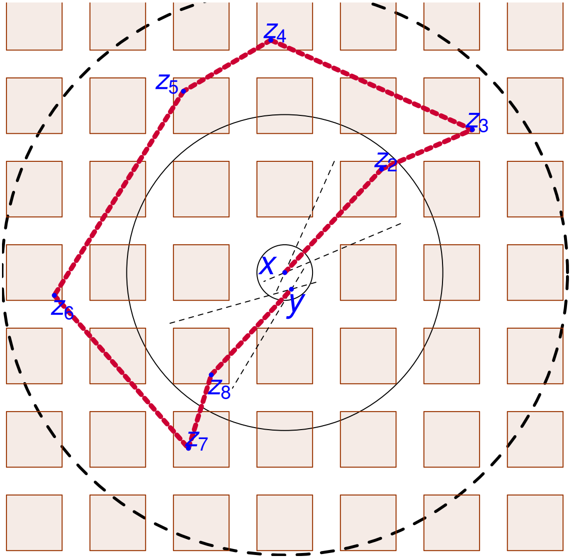
1. *The path p_{xy} starts at x and ends at y.*
2. *Any path p_{xy} has at most N edges.*
3. *Any edge of G is used in at most M paths p_{xy}.*
4. *Any edge used in p_{xy} has length between R_0 and $\lambda|x-y|$.*

Proof based on 2 steps

Step 1: Rescaling resp. renormalization. Connecting blocks of points.



Step 2: Construction of a chain connecting the points *x* and *y*



5. Proof of Theorem 7

We are now in the position to prove Theorem 7, which implies Theorem 1. After all the work above, the proof is just an easy consequence of Theorem 9.

Proof: Let $R > 0$.
For $x, y \in B_R \cap \mathbb{Z}^d$ denote by $(x = z_1, z_2, ..., z_{N-1}, z_N = y)$ the path p_{xy} that satisfies properties (1) – (4) of Theorem 9.

For simplicity, we assume that every path in (p_{xy}) is of length N. Properties (1) – (4) of Theorem 9 can be applied and we conclude . . .

$$\sum_{\substack{x,y\in B_R\cap\mathbb{Z}^d\\|x-y|>R_0}} (f(x)-f(y))^2|x-y|^{-d-\alpha}$$

$$\le 2\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R\cap\mathbb{Z}^d\\|x-y|>R_0}} \sum_{i=1}^{N-1}(f(z_{i+1})-f(z_i))^2|z_{i+1}-z_i|^{-d-\alpha}$$

$$\le 2\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R\cap\mathbb{Z}^d\\|x-y|>R_0}} (N-1) \max_{i\in\{1,\dots,N-1\}} \left[(f(z_{i+1})-f(z_i))^2|z_{i+1}-z_i|^{-d-\alpha}\right]$$

$$\le 2\Lambda\lambda^{d+\alpha} \sum_{\substack{x,y\in B_R\cap\mathbb{Z}^d\\|x-y|>R_0}} (N-1) \max_{i\in\{1,\dots,N-1\}} \left[(f(z_{i+1})-f(z_i))^2\omega(z_{i+1},z_i)\right]$$

$$= 2\Lambda\lambda^{d+\alpha}(N-1)M \sum_{\substack{x,y\in B_{(N-1)\lambda R}\cap\mathbb{Z}^d\\|x-y|>R_0}} (f(x)-f(y))^2\omega(x,y).$$

Set $c = \left(2\Lambda\lambda^{d+\alpha}(N-1)M\right)^{-1}$ and $\kappa = (N-1)\lambda$. □

Motivation and application.

Story III

Boltzmann Equation

We look at the Boltzmann equation with initial data

$$\begin{cases} \frac{\partial f}{\partial t} + \langle v, \nabla_x f \rangle = Q(f,f), \\ f(0,x,v) = f_0(x,v). \end{cases}$$

Here the right hand side is the Boltzmann collision operator

$$Q(g,f)(v) = \int\limits_{\mathbb{R}^d} \int\limits_{S^{d-1}} (g(v_*')f(v') - g(v_*)f(v))B(r,\theta)\, \mathrm{d}\sigma\, \mathrm{d}v_*,$$

where

$$r = |v - v_*|, \quad \cos\theta = \left\langle \sigma, \frac{v - v_*}{r} \right\rangle$$

and (v, v_*) resp. (v', v_*') are velocities of colliding particles before the collision resp. after the collision.
For inverse power law potentials the cross section is given by

$$B(r,\theta) = r^\gamma b(\cos\theta), \quad \gamma > -d,$$

One can decompose $Q(g,f) = Q_1(g,f) + Q_2(g,f)$ by adding and subtracting $g(v'_*)f(v)$ to the integrand of $Q(g,f)$ and splitting the integral. One arrives at

$$\mathbf{Q_1(g,f)}(v) = \int_{\mathbb{R}^d}\int_{S^{d-1}} (f(v') - f(v))\, g(v'_*) B(r,\theta)\, \mathrm{d}\sigma\, \mathrm{d}v_*,$$

$$Q_2(g,f)(v) = f(v)\int_{\mathbb{R}^d}\int_{S^{d-1}} (g(v'_*) - g(v_*))\, B(r,\theta)\, \mathrm{d}\sigma\, \mathrm{d}v_*.$$

Theorem [Silvestre, Comm. Math. Phys. 2016]
For fixed g, the operator $f \mapsto Q_1(f)$ is of the form above. The operator $f \mapsto Q_2(f)$ is of lower order.

Idea of proof:
We write Q_1 in the form

$$\mathbf{Q_1(g,f)}(v) = \int_{\mathbb{R}^d} (f(v') - f(v))\, \mathbf{k_g(v,v')}\, \mathrm{d}v'.$$

and study and study the kernel k_g.

Connection to the Boltzmann Equation

The kernel has the following representation:

$$k_g(v, v') = \frac{2^{2d-1}}{|v' - v|} \int_{\{v'_*:\langle v'_*, v'-v\rangle = 0\}} g(v + v'_*) B(r, \theta) r^{-d+2} \, \mathrm{d}v'_*.$$

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Some properties of the kernel [Silvestre]:

- symmetry corresponding to equations in nondivergence form, i.e. $k_g(v,v+y)=k_g(v,v-y)$
- comparability:

$$k_g(v,v') \asymp \left(\int_{\{v'_*:\langle v'_*, v'-v\rangle=0\}} g(v+v'_*)|v'_*|^{\gamma+\nu+1}\,\mathrm{d}v'_*\right)|v'-v|^{-d-\nu}$$

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- comparability:

$$k_g(v,v') \asymp \left(\int_{\{v'_*:\langle v'_*, v'-v\rangle = 0\}} g(v+v'_*) |v'_*|^{\gamma+\nu+1} \, \mathrm{d}v'_* \right) |v'-v|^{-d-\nu}$$

- lower bound: There are constants $\lambda, \mu > 0$ so that for each $v \in B$ there exists a symmetric subset $A(v) \subset S^{d-1}$ with $|A| \geq \mu$ and

$$k_g(v,v') \geq \lambda |v'-v|^{-d-\nu} \text{ every time } v' \in \underbrace{v + \{r\sigma | r > 0, \sigma \in A\}}_{\text{cone with apex at } v}.$$

Thank you very much

for your attention!