Stochastic recursions

Between Kesten's and Grey's assumptions

Bartosz Kołodziejek

(based on joint work with Ewa Damek)

Warsaw University of Technology

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Będlewo

Stochastic recursions

- $(A_n, B_n)_{n\geqslant 1} \in \mathbb{R} \times \mathbb{R}$ - sequence of i.i.d random vectors, R_0 (resp. M_0) independent of $(A_n, B_n)_{n\geqslant 1}$.

$$R_n = A_n R_{n-1} + B_n, \quad n \geqslant 1$$

and if $A_n \geqslant 0$ a.s

$$M_n = \max\{A_n M_{n-1}, B_n\}, \quad n \geqslant 1.$$

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- Under mild contractivity hypotheses, R_n (resp. M_n) converges in law to a random variable R (resp. M) satisfying

$$\boxed{R \stackrel{d}{=} AR + B}, \qquad R \text{ and } (A, B) \text{ are independent,}$$

resp.

$$\boxed{M \stackrel{d}{=} \max\{AM, B\}}, \qquad M \text{ and } (A, B) \text{ are independent.}$$

Notations and assumptions

Standing Assumptions

- $$\mathbb{P}(A \geqslant 0, B \geqslant 0) = 1$$
- $$\mathbb{E} \log A < 0$$
- *The distribution of* $\log A | A > 0$ *is non-arithmetic.*

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- $\mathcal{R}_0$ - class of slowly varying functions
  $L \in \mathcal{R}_0$ if L measurable and for any $\lambda > 0$,

$$\frac{L(\lambda x)}{L(x)} \to 1 \qquad \text{as } x \to \infty.$$

- $f(x) \sim g(x)$ means $f(x)/g(x) \to 1$ as $x \to \infty$.

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  for some $\alpha > 0$. Then

$$\boxed{\mathbb{P}(R > x) \sim \frac{\mathbb{E}((AR+B)^\alpha - (AR)^\alpha)}{\alpha\rho} \frac{1}{x^\alpha}} \qquad \boxed{\mathbb{P}(M > x) \sim \frac{\mathbb{E}(\max\{AM,B\}^\alpha - (AM)^\alpha)}{\alpha\rho} \frac{1}{x^\alpha}}$$

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$$\mathbb{E}A^\alpha < 1, \qquad \mathbb{E}A^{\alpha+\varepsilon} < \infty \qquad \mathbb{P}(B > x) = \frac{L(x)}{x^\alpha}$$

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$$\boxed{\frac{\widetilde{L}(x)}{L(x)} \to \infty, \qquad \text{as } x \to \infty}$$

Theorem (Main Result)

Under assumptions from the previous slide, one has:

$$x^\alpha \mathbb{P}(M > x) \sim x^\alpha \mathbb{P}(R > x) \sim \frac{\widetilde{L}(x)}{\rho} = \frac{\mathbb{E} B^\alpha I_{B \leqslant x}}{\alpha \rho}.$$

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If additionally the law of A has nontrivial absolutely continuous component, then

$$x^\alpha \mathbb{P}(M > x) = \frac{\widetilde{L}(x)}{\rho} - \frac{\mathbb{E} \min\{AM, B\}^\alpha}{\alpha \rho} + O(L(x)).$$

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Assume further that there exists $\beta > 0$ such that

$$\limsup_{h \to 0^+} \sup_{a \in \mathbb{R}} h^{-\beta} \mathbb{P}(a < \log A \leqslant a + h) < \infty$$

and $\mathbb{E} A^\gamma < \infty$ for some $\gamma > \max\{\alpha, \alpha^2/\beta\}$. Then

$$x^\alpha \mathbb{P}(R > x) = \frac{\widetilde{L}(x)}{\rho} + \frac{\mathbb{E}\left((AR + B)^\alpha - (AR)^\alpha - B^\alpha\right)}{\alpha \rho} + O(L(x)) + o(1).$$

Sketch of proof 1

- $f(x) = e^{\alpha x}\mathbb{P}(R > e^x)$, $\quad$ $[resp. R \mapsto M]$
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- $S_k = Z_1 + \ldots + Z_k$, $(Z_k)_k$ i.i.d.

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- $S_k = Z_1 + \ldots + Z_k$, $(Z_k)_k$ i.i.d.
- We use Goldie's approach leading to implicit renewal equation:

$$\begin{aligned} f(x) &= \psi(x) + e^{\alpha x}\mathbb{P}(R > e^{x-\log A}) = \psi(x) + \mathbb{E}A^\alpha f(x - \log A) \\ &= \psi(x) + \mathbb{E}f(x - Z_1) = \ldots \\ &= \sum_{k=0}^{n-1} \mathbb{E}\psi(x - S_k) + \mathbb{E}f(x - S_n) \xrightarrow{n} \sum_{k=0}^{\infty} \mathbb{E}\psi(x - S_k) \\ &= \int_{\mathbb{R}} \psi(x-t)\, dH_Z(t), \end{aligned}$$

where H_Z is the *renewal function* defined by

Definition

$$H_Z(t) = \sum_{k=0}^{\infty} \mathbb{P}(S_k \leqslant t).$$

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- If ψ is dRi (**almost** the case under KGG's assumptions), then

$$\int_{\mathbb{R}} \psi(x-t)dH_Z(t) \xrightarrow{x\to\infty} \frac{1}{\mathbb{E}Z}\int_{\mathbb{R}} \psi(t)dt$$

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- Define $\psi_B(x) = e^{\alpha x}\mathbb{P}(B > e^x) = L(e^x)$. Then

$$\psi = \psi_B + \psi_0,$$

where ψ_0 is **almost** dRi, and thus we have **almost**

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- It remains to work out the asymptotics of the main part, that is

$$\int_{\mathbb{R}} L(e^{x-t})dH_Z(t)$$

Main technical result

Theorem (Renewal Theorem)

Assume

- $\mathbb{E}Z \in (0,\infty)$, *the distribution of* Z *is non-arithmetic,* $\mathbb{P}(Z \leqslant x) = o(e^{rx})$ *as* $x \to -\infty$
- $L(x) = x^{\alpha}\mathbb{P}(B > x) \in \mathcal{R}_0$.

Then,

$$\int_{\mathbb{R}} L(e^{x-t})dH_Z(t) \sim \int_{(0,x]} L(e^{x-t})dH_Z(t) \sim \frac{1}{\mathbb{E}Z}\widetilde{L}(e^x).$$

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If additionally

- $\limsup_{|t|\to\infty} |\mathbb{E}e^{itZ}| < 1$ *and* $\mathbb{E}e^{\varepsilon Z} < \infty$ *for some* $\varepsilon > 0$,

then

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- Following Vervaat, R satisfies another stochastic equation

$$\boxed{R \overset{d}{=} A_1 \dots A_N R + B_N^*}, \qquad R \text{ and } (A_1 \dots A_N, R_N^*) \text{ are independent}$$

where $B_n^* := B_1 + A_1 B_2 + \ldots + A_1 \dots A_{n-1} B_n$ for $n \geqslant 1$.

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Goal: describe properties of $\Pi_N := A_1 \dots A_N$ and B_N^*.

Proposition (New?)

$$B_N^* \overset{d}{=} (-A_-)S + B, \qquad S \text{ and } (A_-, B) \text{ are independent}$$

and S satisfies

$$S \overset{d}{=} A_+ S + B, \qquad S \text{ and } (A_+, B) \text{ are independent.}$$

Proof: Since $\{N \geqslant k\} = \{A_1 < 0, A_2 > 0, \ldots, A_{k-1} > 0\}$ for $k \geqslant 2$ we have

$$\begin{aligned} B_N^* &= \sum_{k=1}^{\infty} I_{N \geqslant k} A_1 \ldots A_{k-1} B_k \\ &= B_1 + \sum_{k=2}^{\infty} I_{A_1 < 0} I_{A_2 > 0} \ldots I_{A_{k-1} > 0} A_1 A_2 \ldots A_{k-1} B_k \\ &= B_1 - (A_1)_- \left(\sum_{k=2}^{\infty} (A_2)_+ \cdots (A_{k-1})_+ B_k \right). \end{aligned}$$

Our setup

$$N := \inf\{n\colon A_1 \dots A_n \geqslant 0\}$$

$$\Pi_N = A_1 \dots A_N$$

$$B_N^* = B_1 + A_1 B_2 + \dots + A_1 \dots A_{N-1} B_N$$

Theorem

Under our setup

- $$\mathbb{E}\Pi_N^\alpha = 1 \quad \textit{and} \quad \mathbb{E}\Pi_N^\alpha \log \Pi_N = 2\mathbb{E}|A| \log |A|.$$
- $$\mathbb{P}(B_N^* > t) \sim \mathbb{P}(|B| > t).$$

$$R \stackrel{d}{=} AR + B$$

Theorem

Suppose that

(A1) $\mathbb{P}(A < 0) > 0$, $\mathbb{E}\log|A| < 0$,

(A2) $\mathbb{E}|A|^\alpha = 1$, $\mathbb{E}|A|^{\alpha+\varepsilon} < \infty$

(A3) *the distribution of* $\log|A|$ *given* $|A| > 0$ *is non-arithmetic,*

(B1) $p + q = 1$, $L \in \mathcal{R}_0$

$$\mathbb{P}(B > x) \sim p\frac{L(x)}{x^\alpha}, \qquad \mathbb{P}(B < -t) \sim q\frac{L(x)}{x^\alpha},$$

(B2) $\mathbb{E}|B|^\alpha = \infty$.

Then

$$x^\alpha \mathbb{P}(R > x) \sim \frac{\widetilde{L}(x)}{2\rho}.$$