Stability of heat kernel estimates for jump diffusions under Feynman-Kac perturbations

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Schrödinger operator and local Feynman-Kac transform

$$\begin{aligned}
\Delta/2 &\leftrightarrow \text{Brownian motion}, && T_t f(x) = \mathbb{E}_x[f(B_t)];\\
\Delta/2 + q &\leftrightarrow \text{F-K transform}, && T_t f(x) = \mathbb{E}_x[e^{\int_0^t q(B_s)ds} f(B_t)];\\
\Delta/2 + \mu &\leftrightarrow \text{F-K transform}, && T_t f(x) = \mathbb{E}_x[e^{A_t^\mu} f(B_t)].
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Hunt process X, with infinitesimal generator $\mathcal{L}$.

$$\mathcal{L} + \mu \leftrightarrow \text{ F-K transform, } T_t f(x) = \mathbb{E}_x[e^{A_t^\mu} f(X_t)].$$

Non-local Feynman-Kac transform

Recall that a Hunt process X_t admits a Lévy system $(N(x, dy), H_t)$, if for any $x \in \mathbb{R}^d$, any stopping time T and any non-negative measurable function φ on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, vanishing on the diagonal,

$$\mathbb{E}_x \left[\sum_{s \le T} \varphi(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}^d} \varphi(s, X_s, y) N(X_s, dy) dH_s \right]. \quad (1)$$

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For example, a symmetric α-stable process has a Lévy system as follows: $N(x,dy)=\frac{1}{|x-y|^{d+\alpha}}dy$ and $H_t=t$.

Non-local Feynman-Kac transform

For discontinuous process X, one can perform a non-local Feynman-Kac transform

$$T_t^{\mu,F} f(x) := \mathbb{E}_x \left[\exp\left(A_t^{\mu} + \sum_{0<s\le t} F(X_{s-}, X_s) \right) f(X_t) \right].$$

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The infinitesimal generator for $\{T_t^{\mu,F}; t \geq 0\}$ is

$$\mathcal{L}^{\mu,F} := \mathcal{L} + \mu + \mu_H \mathbf{F},$$

where μ_H is the Revuz measure of the PCAF H and

$$\mu_H \mathbf{F} f(dx) := \left(\int_E (e^{F(x,y)} - 1) f(y) N(x, dy) \right) \mu_H(dx).$$

Stability of heat kernel estimates under F-K perturbation

- Blanchard-Ma (1990) showed that under a certain Kato class condition, the heat kernel of a local Feynman-Kac semigroup of Brownian motion admits two-sided Gaussian bound estimates.
- Song (2006) showed that under a certain Kato class condition, the heat kernel of the non-local Feynman-Kac semigroup of a symmetric stable-like process X on $\mathbb{R}^d$ is comparable to that of X.
- Chen-Kim-Song (2014) showed that Dirichlet heat kernel estimates for a class of (not necessarily symmetric) Markov processes are stable under non-local Feynman-Kac perturbations.

In our study, we start with a Hunt process X on $\mathbb{R}^d$ that has a jointly continuous transition density function $p(t,x,y)$ that enjoys two-sided estimates, there exist constants $C \geq 1$ such that for $0 < t \leq 1$ and $x, y \in \mathbb{R}^d$,

$$C^{-1}\left(\Gamma_{c_2}(t;x-y) + \eta(t;x-y)\right) \leq p(t,x,y) \leq C\left(\Gamma_{c_4}(t;x-y) + \eta(t;x-y)\right), \tag{2}$$

where

$$\Gamma_c(t;x) := t^{-d/2} e^{-c|x|^2/t} \quad \text{and} \quad \eta(t;x) := \frac{t}{(t^{1/2} + |x|)^{d+\alpha}}.$$

Jump diffusions: symmetric

symmetric diffusion processes with jumps (Chen-Kumagai, 2010)

$$\mathcal{L}u(x) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\Big) + \lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}}(u(y)-u(x))\frac{c(x,y)}{|x-y|^{d+\alpha}}dy,$$

where $\alpha \in (0,2)$, $(a_{ij}(x))_{1\le i,j\le d}$ is uniformly elliptic and bounded, and $c(x,y)$ is symmetric and $c_1 \le c(x,y) \le c_2$.
It is shown that there is a Feller process X having strong Feller property associated with $\mathcal{L}$.

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Chen-Kumagai showed that there are two-sided heat kernel estimates for X, and they can be rewritten into the compact form (2) on fixed time intervals.

Main Result

Theorem (Chen-W. 2017)

Suppose X has transition density function $p(t, x, y)$ with the two-sided heat kernel estimates (2) holds on $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$. Let μ and $F_1 = e^F - 1$ satisfy certain condtions, then the non-local Feynman-Kac semigroup $(T_t^{\mu,F}; t \geq 0)$ has a jointly continuous kernel $q(t, x, y)$ so that $T_t^{\mu,F} f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) dy$ for every bounded Borel measurable function f on $\mathbb{R}^d$. Moreover, there exist positive constants $\tilde{c}_3$, K so that for any $t > 0$

$$q(t, x, y) \leq \tilde{c}_3 e^{Kt} \left(\Gamma_{2c_4/3}(t; x - y) + \eta(t; x - y) \right).$$

If in addition, F satisfies certain condtion, then there exist positive constants $\tilde{c}_1$, λ_1 and K_1 so that for any $t > 0$ and $x, y \in \mathbb{R}^d$,

$$q(t, x, y) \geq \tilde{c}_1 e^{-K_1 t} \left(\Gamma_{\lambda_1}(t; x - y) + \eta(t; x - y) \right).$$

Set-up

In the study of non-local Feynman-Kac perturbation, it is convenient to use Stieltjes exponential $\mathrm{Exp(K)t}$ of $K_t := A_t^\mu + \sum_{s\le t} F_1(X_{s-}, X_s)$, where $F_1 = e^F - 1$. $\mathrm{Exp(K)_t}$ is the unique souliton of $Z_t = 1 + \int_{(0,t]} Z_{s-} dK_s$ and

$$\exp(A_t^\mu + \sum_{s\le t} F(X_{s-}, X_s)) = \mathrm{Exp}(K)_t = e^{K_t^c} \prod_{0\le s\le t} (1 + \Delta K_s).$$

Then we can express the semigroup into infinite sum of Lebesgue-Stieltjes integrals:

$$T_t^{\mu,F} f(x) = P_t f(x) + \mathbb{E}_x \left[f(X_t) \sum_{n=1}^{\infty} \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right].$$

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The advantage of using Stieltjes exponential is the above identity that allows one to use Markov property.

Set-up

Using Markov property of X, we get

$$T_t^{\mu,F} f(x) = P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \Big[\int_{[0,t)} \Big(\mathbb{E}_{X_{s_{n-1}}} \Big[\int_{(0,t-s_{n-1}]} P_{t-s_{n-1}-r} f(X_r) dK_r \Big] \times \int_{[0,s_{n-1})} dK_{s_{n-2}} \cdots \int_{[0,s_2)} dK_{s_1} \Big) dK_{s_{n-1}} \Big].$$

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$$T_t^{\mu,F}f(x) = P_tf(x) + \sum_{n=1}^{\infty}\mathbb{E}_x\Bigg[\int_{[0,t)}\Bigg(\mathbb{E}_{X_{s_{n-1}}}\Bigg[\int_{(0,t-s_{n-1}]}P_{t-s_{n-1}-r}f(X_r)dK_r\Bigg] \times \int_{[0,s_{n-1})}dK_{s_{n-2}}\cdots\int_{[0,s_2)}dK_{s_1}\Bigg)dK_{s_{n-1}}\Bigg].$$

For any bounded $g \geq 0$,

$$\begin{aligned}
&\mathbb{E}_x\left[\int_{(0,s]}g(s-r,X_r)dK_r\right] \\
&= \int_0^s\int_{\mathbb{R}^d}p(r,x,y)g(s-r,y)\mu(dy)dr \\
&+ \int_0^s\int_{\mathbb{R}^d}p(r,x,y)\left(\int_{\mathbb{R}^d}g(s-r,y)F_1(z,y)\frac{c(z,y)}{|z-y|^{d+\alpha}}dy\right)dzdr.
\end{aligned}$$

Set-up

We define $p^{(0)}(t,x,y) := p(t,x,y)$, and for $k \geq 1$,

$$p^{(k)}(t,x,y) := \int_0^t \left(\int_{\mathbb{R}^d} p(t-s,x,z)p^{(k-1)}(s,z,y)\mu(dz) \right) ds$$
$$+ \int_0^t \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s,x,z)p^{(k-1)}(s,w,y) \frac{c(z,w)F_1(z,w)}{|z-w|^{d+\alpha}} dzdw \right) ds.$$

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If $\sum_{k=0}^{\infty} p^{(k)}(t,x,y)$ converges to $q(t,x,y)$, then $q(t,x,y)$ is the heat kernel for the semigroup $\{T_t^{\mu,F}; t \geq 0\}$.

Kato classes

Definition (Kato classes)

(i) A signed measure μ on $\mathbb{R}^d$ is said to be in the Kato class $\mathbf{K}_\alpha$ if

$$\begin{cases} \lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{B(x,r)} |x-y|^{(1+\alpha)/3}|\mu|(dy)=0 & d=1; \\ \lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{B(x,r)} \ln\frac{1}{|x-y|}|\mu|(dy)=0 & d=2; \\ \lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{B(x,r)} |x-y|^{2-d}|\mu|(dy)=0 & d\geq 3; \end{cases} \tag{3}$$

(ii) A bounded measurable function F on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on the diagonal, is said to be Kato class $\mathbf{J}_\alpha$ if the above condition holds for $\left(\int_{\mathbb{R}^d}\frac{|F(y,w)|+|F(w,y)|}{|y-w|^{d+\alpha}}dw\right)dy$ in place of $|\mu|(dy)$.

Upper bound estimates

Denote by

$$N_{\mu}^{\alpha,\lambda}(t) := \sup_{x\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \left(\Gamma_\lambda(s; x-y) + \eta(s; x-y)\right) |\mu|(dy)ds.$$

$$N_{F}^{\alpha,\lambda}(t) := \sup_{y\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\Gamma_\lambda(s; y-z) + \eta(s; y-z)\right) \frac{|F(z,w)| + |F(w,z)|}{|z-w|^{d+\alpha}} dwdzds.$$

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Lemma

For any $k \geq 0$ and $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$, there exists a constant M such that

$$|p^{(k)}(t,x,y)| \leq Cp_{2c_4/3}(t,x,y)\Big((MN_{\mu,F_1}^{\alpha,c_4/3}(t))^k + k\|F_1\|_\infty M(MN_{\mu,F_1}^{\alpha,c_4/3}(t))^{k-1}\Big)$$

Upper bound estimate

Theorem (Chen-W, 2017)

The series $\sum_{k=0}^{\infty} p^{(k)}(t,x,y)$ converges absolutely to a jointly continuous function $q(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$. The function $q(t,x,y)$ is the integral kernel (or, heat kernel) for the Feynman-Kac semigroup $\{T_t^{\mu,F}; t\geq 0\}$, and there exist constants $\tilde{c}_3, K$ depending on $d,\alpha,\|F_1\|_\infty, N_{\mu,F_1}^{\alpha,c_4/3}, C, c_4$ such that

$$q(t,x,y)\leq \tilde{c}_3 e^{Kt} p_{2c_4/3}(t,x,y) \quad \text{for every } (t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d.$$

Lower bound estimate

Near-diagonal estimate

We assume in addition $F \in \mathbf{J}_\alpha$, there exist constants $K_2, \lambda_1 > 0$ so that

$$q(t,x,y) \geq K_2 t^{-d/2} \exp\left(-\frac{\lambda_1 |x-y|^2}{t}\right) \text{ for } (t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d.$$

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This is the Gaussian component in the lower bound estimate. It dominates the estimate when $|x-y| \leq \sqrt{t}$.

Lower bound estimate: jumping component

We consider a sub-Markovian semigroup $\{Q_t; t \geq 0\}$,

$$Q_t f(x) := \mathbb{E}_x \left[\exp \left(-A_t^{|\mu|} - \sum_{s \leq t} |F|(X_{s-}, X_s) \right) f(X_t) \right].$$

There exists a Feller process Y such that $Q_t f(x) = \mathbb{E}_x[f(Y_t)]$.

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With $\mu \in \mathbf{K}_\alpha, F \in \mathbf{J}_\alpha$, by Jensen's inequality,

$$\inf_{x \in \mathbb{R}^d} \mathbb{E}_x \left[\exp(A^{-|\mu|, -|F|})_1 \right] \geq \exp\left(- \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[A_1^{|\mu|, |F|} \right] \right) =: \gamma_0 > 0.$$

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Define the random time η with distribution $\mathbb{P}_x(\eta > t) = \mathbb{E}_x \left[\exp(A^{-|\mu|,-|F|})_t \right]$, we can now couple (X, Y) such that on $\{\eta > t\}$, $Y_s = X_s$ for every $s \leq t$.

Lemma

There exists a constant $\kappa_0 \in (0, 1)$ depending on $d, C, c_4, \alpha, \gamma_0$ such that for any $0 < r \leq 1$,

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}_x \left(\tau^X_{B(x,r)} \leq \kappa_0 r^2 \right) \leq \gamma_0/2.$$

Consequently, for every $x \in \mathbb{R}^d$ and $r \in (0, 1]$,

$$\mathbb{P}_x \left(\tau^Y_{B(x,r)} > \kappa_0 r^2 \right) \geq \gamma_0/2. \tag{4}$$

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Lemma

There exists a constant $\gamma_1 > 0$ so that for any $r > 0$ and $x_0, y_0 \in \mathbb{R}^d$ with $|y_0 - x_0| \ge 3r$,

$$\mathbb{P}_{x_0}\left(\sigma^Y_{B(y_0,r)} \le \kappa_0 r^2\right) \ge \gamma_1 \frac{r^{d+2}}{|y_0 - x_0|^{d+\alpha}}. \tag{5}$$

Two-sided estimates

Theorem (Chen-W, 2017)

There exist positive constants $\widetilde{K} \geq 1$ and $\lambda_1 > 0$ depending on $d, \alpha, N_{\alpha,F}^{\alpha,c_4/3}, \|F\|_\infty$ and the constants in (2) such that

$$\widetilde{K}^{-1} p_{\lambda_1}(t,x,y) \leq q(t,x,y) \leq \widetilde{K} p_{2c_4/3}(t,x,y) \tag{6}$$

for $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$.

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Sketch of proof: it suffices to consider that $|x-y| > \sqrt{t}$, set $r = \sqrt{t}/3$,

$$\begin{aligned}\int_{B(y,2r)} q(2\kappa_0 r^2, x, z)dz \geq& \mathbb{P}_x(Y_{2\kappa_0 r^2} \in B(y,2r)) \\ \geq& \mathbb{P}_x\left(\tau^Y_{B(x,r)} > \kappa_0 r^2\right) \mathbb{P}_x\left(\sigma^Y_{B(y,r)} < \kappa_0 r^2\right),\end{aligned}$$

$$q(t,x,y) \geq \inf_{z\in B(y,2r)} q(t-2\kappa_0 r^2, y, z) \int_{B(y,2r)} q(2\kappa_0 r^2, x, z) \geq K\eta(t; x-y)$$

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Thank you!