Optimal uniform approximation of Lévy processes on Banach spaces with finite variation processes

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Optimisation problem, setting

- X_t, $t \geq 0$, is a càdlàg Lévy process attaining its values in a Banach space V (i.e. a process with a.s. càdlàg paths and independent and stationary increments).
- $\mathcal{A}_X$ is a family of V-valued processes Y_t, $t \geq 0$, adapted to the natural filtration of X.
- $|\cdot|$ denotes the norm in V and for $T > 0$ and two processes $Y, Z : \Omega \times \mathcal{T} \to V$, where $\mathcal{T}$ is an index set such that $[0, T] \subset \mathcal{T}$, we denote
$$\|Y - Z\|_{\infty,[0,T]} := \sup_{0 \leq t \leq T} |Y_t - Z_t|$$
and
$$\text{TV}(Y, [0, T]) := \sup_n \sup_{0 \leq t_0 < t_1 < \cdots < t_n \leq T} \sum_{i=1}^{n} \left| Y_{t_i} - Y_{t_{i-1}} \right|.$$

Optimisation problem, formulation

We will deal with the following optimisation problem.
Given are $T, \theta > 0$ and non-decreasing function $\psi : [0, +\infty) \to [0, +\infty)$ calculate (or estimate up to universal constants)

$$V_X(\psi, \theta) := \mathbb{E} \inf_{Y \in \mathcal{A}_X} \left\{ \psi\left(\|X - Y\|_{\infty,[0,T]} \right) + \theta \cdot \mathrm{TV}(Y, [0, T]) \right\}. \qquad (1)$$

To make the problem non-trivial we assume that $\mathbb{E}|X_1| < +\infty$.

Remark

This type of optimisation problems appears naturally in several stituations. However, it has no unified, algorithmic solution since the generator of the total variation functional is not well defined. Moreover, we deal with very general Lévy processes attaining their values in general Banach spaces.

Optimisation problem - first observations

From the triangle inequality we immediately get that

$$\|X - Y\|_{\infty,[0,T]} \leq c/2 \Rightarrow |Y_t - Y_s| \geq \max\{|X_t - X_s| - c, 0\}$$

for any $0 \leq s \leq t \leq T$.
Thus

$$\begin{aligned} \mathrm{TV}(Y, [0, T]) &\geq \mathrm{TV}^c(X, [0, T]) \\ &:= \sup_n \sup_{0 \leq t_0 < t_1 < \cdots < t_n \leq T} \sum_{i=1}^n \max\{|X_t - X_s| - c, 0\}. \end{aligned} \tag{2}$$

The quantity on the right side of (2) is called the **truncated variation** of X.

Optimisation problem - alternative statement

From the results of [LochowskiMilosSPA], [LochowskiGhomrasniMMAS] it is possible to prove that for any $c > 0$ there exists a process $X^c \in \mathcal{A}_X$ such that $\|X - X^c\|_{\infty,[0,T]} \le c/2$ and

$$\mathrm{TV}^c(X,[0,T]) \le \mathrm{TV}(X^c,[0,T]) \le \mathrm{TV}^c(X,[0,T]) + c, \qquad (3)$$

thus in the case $V = \mathbb{R}$ we have the estimate

$$\begin{aligned} &\inf_{c>0}\left\{\psi\left(\frac{c}{2}\right) + \theta\cdot\mathbb{E}\mathrm{TV}^c(X,[0,T])\right\} \\ &\le \mathbb{E}\inf_{Y\in\mathcal{A}_X}\left\{\psi\left(\|X-Y\|_{\infty,[0,T]}\right) + \theta\cdot\mathrm{TV}(Y,[0,T])\right\} \\ &\le \inf_{c>0}\left\{\psi\left(\frac{c}{2}\right) + \theta\cdot\mathbb{E}\mathrm{TV}^c(X,[0,T]) + \theta c\right\}, \end{aligned}$$

which means that if $\psi(x)$ grows no faster than some polynomial and no slower than a linear function then both quantities $\inf_{c>0}\{\psi(c/2) + \theta\cdot\mathbb{E}\mathrm{TV}^c(X,[0,T])\}$ and $V_X(\psi,\theta)$ are comparable up to universal constants depending on θ and ψ only.

Optimisation problem - some results for real Brownian motion with drift

Unfortunately, the quantity $\mathrm{TV}^c(X, [0, T])$ is still not easy one to calculate/estimate.
In [LochowskiBPAS] the following estimates of $\mathbb{E}\mathrm{TV}^c(X, [0, T])$ were given for standard Brownian motion with drift $W_t = B_t + \mu t$:

- for T such that $\sqrt{T} \geq \chi(c, \mu)$,

$$\frac{1}{264}\left(\frac{1}{c} + |\mu|\right) T \leq \mathbb{E}\, TV^c(W, [0; T]) \leq 64\left(\frac{1}{c} + |\mu|\right) T;$$

- for T such that $c - |\mu|\, T \leq \sqrt{T} < \chi(c, \mu)$,

$$\frac{1}{747}\left(2\sqrt{T} + |\mu|\, T - c\right) \leq \mathbb{E}\, TV^c(W, [0; T]) \leq 340\left(2\sqrt{T} + |\mu|\, T - c\right)$$

and for T such that $\sqrt{T} < c - |\mu|\, T$,

$$\frac{1}{227} T^{3/2} \frac{e^{-(c-|\mu| T)^2/(2T)}}{(c - |\mu|\, T)^2} \leq \mathbb{E}\, TV^c(W, [0; T]) \leq 493 \cdot T^{3/2} \frac{e^{-(c-|\mu| T)^2/(2T)}}{(c - |\mu|\, T)^2}.$$

The problem gets even worse in the Banach space setting

The problem gets even worse for more general Lévy processes.
Moreover, in the Banach space setting (even in $\mathbb{R}^2$) the estimate (3) is no longer valid.
Fortunately, we have the following (not difficut to obtain) result:

Theorem (Banach space estimate)

For any $c > 0$ and any regulated process X there exists a process $Y^c \in \mathcal{A}_X$ such that $\|X - Y^c\|_{\infty,[0,T]} \leq c/2$ and

$$\mathbb{E}TV^c(X,[0,T]) \leq \mathbb{E}TV(Y^c,[0,T]) \leq \inf_{\lambda>1} \lambda \cdot \mathbb{E}TV^{(\lambda-1)\cdot c/(2\lambda)}(X,[0,T]).$$

Banach space estimate

From the Banach space estimate theorem and assuming that for any $a \geq 0$, $\psi(2a) \leq K_\psi \cdot \psi(a)$, we get

$$\begin{aligned}
&\inf_{c>0}\left\{\psi\left(\frac{c}{2}\right)+\theta\cdot\mathbb{E}\mathrm{TV}^c(X,[0,T])\right\}\\
&\leq \inf_{c>0}\left\{\psi\left(\frac{c}{2}\right)+\theta\cdot\mathbb{E}\mathrm{TV}(Y^c,[0,T])\right\}\\
&\leq \max\left(K_\psi^2,2\right)\inf_{c>0}\left\{\psi\left(\frac{c}{2}\right)+\theta\cdot\mathbb{E}\mathrm{TV}^c(X,[0,T])\right\}
\end{aligned} \tag{4}$$

thus again we see that both quantities $\inf_{c>0}\left\{\frac{c}{2}+\theta\cdot\mathbb{E}\mathrm{TV}^c(X,[0,T])\right\}$ and $V_X(\psi,\theta)$ are comparable up to universal constants (depending on ψ only).

Banach space estimate - construction

In the case when X has càdlàg trajectories, the construction of the process Y^c simplifies to the following one. First, we define stopping times $\tau_0^c = 0$ and for $n = 1, 2, \ldots$

$$\tau_n^c = \begin{cases} \inf\left\{t > \tau_{n-1}^c : \left|X_{\tau_{n-1}^c} - X_t\right| > \frac{c}{2}\right\} & \text{if } \tau_{n-1}^c < +\infty; \\ +\infty & \text{if } \tau_{n-1}^c = +\infty \end{cases}$$

and then we define

$$Y_t^c = \sum_{n=0}^{+\infty} X_{\tau_n^c} \mathbf{1}_{\left[\tau_n^c; \tau_{n+1}^c\right)}(t).$$

Banach space estimate expressed in simple quantities

Theorem

Let $\psi : [0,+\infty) \to [0,+\infty)$ *be a non-decreasing function such that for* $a \geq 0$, $\psi(2a) \leq K_\psi \cdot \psi(a)$. *For any* $T, \theta > 0$ *the following estimates hold:*

$$\mathbb{E} \inf_{Y \in \mathcal{A}_X} \left\{ \psi \left(\|X - Y\|_{\infty,[0,T]} \right) + \theta \cdot TV(Y,[0,T]) \right\}$$

$$\leq \inf_{c>0} \left\{ \psi \left(\frac{c}{2} \right) + \theta \cdot e \frac{\mathbb{E}\left(|X_{\tau^c} - X_0| \mathbf{1}_{\{\tau^c \leq T\}} \right)}{1 - \mathbb{E} \exp\left(-\tau^c / T \right)} \right\},$$

$$\mathbb{E} \inf_{Y \in \mathcal{A}_X} \left\{ \psi \left(\|X - Y\|_{\infty,[0,T]} \right) + \theta \cdot TV(Y,[0,T]) \right\}$$

$$\geq \frac{1}{\max\left(K_\psi^2, 2 \right)} \inf_{c>0} \left\{ \psi \left(\frac{c}{2} \right) + \theta \cdot \frac{e-1}{2e} \frac{\mathbb{E}\left(|X_{\tau^c} - X_0| \mathbf{1}_{\{\tau^c \leq T\}} \right)}{1 - \mathbb{E} \exp\left(-\tau^c / T \right)} \right\},$$

where $\tau^c = \inf\{ t > 0 : |X_t - X_0| > c/2 \}$.

Some special cases - the case of a Brownian motion with drift

The easiest case is the case of the standard Brownian. Slightly more complicated is the case of the Brownian motion with drift.
Using the already presented estimates for the Brownian motion with drift or a little bit refined reasoning we get

$$V_X(\psi,\theta) = \kappa_2 \inf_{c>0}\left\{\psi\left(\frac{c}{2}\right) + \theta\cdot c\frac{1-\Phi\left(\left(\frac{c}{2}-|\mu|\,T\right)/\sqrt{T}\right)}{1-\frac{2\cosh\left(\frac{c\mu}{2}\right)\sinh\left(\frac{c}{2}\sqrt{\frac{2}{T}+\mu^2}\right)}{\sinh\left(c\sqrt{\frac{2}{T}+\mu^2}\right)}}\right\},$$

where $\kappa_2 \in \left[\frac{e-1}{14e\max\left(K_\psi^2,2\right)}, 4e\right]$ and $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}\mathrm{d}t$.

Some special cases - the case of a standard Brownian motion on $\mathbb{R}^n$

$$V_X(\psi,\theta) = \kappa_3 \inf_{c>0}\left\{\psi\left(\frac{c}{2}\right) + \theta\cdot\frac{c}{2}\,\frac{1-\frac{2^{1-\nu}}{\Gamma(\nu+1)}\sum_{k=1}^{+\infty}\frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}\left(j_{\nu,k}\right)}e^{-2j_{\nu,k}^2 T/c^2}}{1-\frac{(c/2)^{\nu}}{\Gamma(\nu+1)(2T)^{\nu/2}I_{\nu}\left(c\sqrt{1/(2T)}\right)}}\right\},$$

where $\kappa_3 \in \left[\frac{e-1}{2e\max\left(K_{\psi}^2,2\right)}, e\right]$, J_ν denotes the Bessel function of the first kind,

$$J_\nu(y) = \left(\frac{y}{2}\right)^{\nu}\sum_{m=0}^{+\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m}$$

and $0 < j_{\nu,1} < j_{\nu,2} < \ldots$ denote all positive zeros of the function J_ν, and I_ν denotes the modified Bessel function

$$I_\nu(y) = \left(\frac{y}{2}\right)^{\nu}\sum_{m=0}^{+\infty}\frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m}.$$

The case of a standard Brownian motion on $\mathbb{R}^n$ - remarks

The formula presented on the previous slide is really hard to apply.

However, if we fix n or allow to depend the accuracy of our estimate on n then one may apply the results of Grzegorz Serafin proved his recent paper *Exit times densities of the Bessel process*, Proc. Amer. Math. Soc., to appear, DOI: 10.1090/proc/13419.

Special case - real, discontinuous Lévy processes

In this case we have discontinuities and thus we need to take into account overshoots while estimating

$$\mathbb{E}\left(\left|X_{\tau^c} - X_0\right| \mathbf{1}_{\{\tau^c \le T\}}\right).$$

We have the following Quintuple Law at First passage of Doney and Kyprianou:

$$\mathbb{P}\left(\tau^c - \bar{G}_{\tau^c-} \in \mathrm{d}t, \bar{G}_{\tau^c-} \in \mathrm{d}s, X_{\tau^c} - c \in \mathrm{d}u, x - X_{\tau^c-} \in \mathrm{d}v, x - \bar{X}_{\tau^c-} \in \mathrm{d}y\right)$$
$$= \mathcal{U}(\mathrm{d}s, x - \mathrm{d}y)\hat{\mathcal{U}}(\mathrm{d}t, \mathrm{d}v - y)\Pi(\mathrm{d}u + v),$$

where

- $\bar{G}_{\tau^c-}$ is the time of the last maximum prior to first passage,
- $\tau^c - \bar{G}_{\tau^c-}$ is the length of the excursion making first passage,
- $X_{\tau^c} - c$ is the overshoot at first passage,
- $x - X_{\tau^c-}$ is the undershoot at first passage,
- $x - \bar{X}_{\tau^c-}$ is the undershoot of the last maximum at first passage.

Special case - real, discontinuous Lévy processes, cont.

Finally, $\mathcal{U}$ and $\hat{\mathcal{U}}$ are bivariate potential measures of the ascending ladder process (L^{-1}, H) and the decending ladder process $(\hat{L}^{-1}, \hat{H})$ respectively:

$$\mathcal{U}(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \mathrm{d}t \mathbb{P}\left(L^{-1} \in \mathrm{d}s, H \in \mathrm{d}x\right)$$

$$\hat{\mathcal{U}}(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \mathrm{d}t \mathbb{P}\left(\hat{L}^{-1} \in \mathrm{d}s, \hat{H} \in \mathrm{d}x\right).$$

Unfortunately, these quantities are computable for very special families of processes (even in the α-stable case).

Some references

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Thank you for your attention!