Exotic semigroups associated with classical orthogonal expansions and their maximal operators

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General motivation

Classical orthogonal families, e.g.

- **Laguerre polynomials** $\{L_n^\alpha : n \in \mathbb{N}^d\}$
  OgB in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$ iff $\alpha \in (-1,\infty)^d$
  associated 'Laplacian' L_α, 'heat' semigroup $\exp(tL_\alpha)$, etc.
- **Jacobi polynomials** $\{P_n^{\alpha,\beta} : n \in \mathbb{N}^d\}$
  OgB in $L^2((-1,1)^d, d\rho_{\alpha,\beta})$ iff $\alpha, \beta \in (-1,\infty)^d$
  associated 'Laplacian' $L_{\alpha,\beta}$ and related objects
- **continuous Fourier-Bessel system** $\{\varphi_z^\nu : z \in \mathbb{R}_+^d\}$
  defines the Hankel transform in $L^2(\mathbb{R}_+^d, d\eta_\nu)$ iff $\nu \in (-1,\infty)^d$
  associated 'Laplacian' L_ν, ...

Q: what if the parameters are **exotic**, i.e. not in the above ranges?
[keep the differential 'Laplacians' (and the measure spaces),
are there orthogonal systems leading to self-adjoint extensions?]

Classical Laguerre setting (1-dim)

Laguerre 'Laplacian'

$$L_\alpha = -x\frac{d^2}{dx^2} - (\alpha+1-x)\frac{d}{dx}$$

- symmetric in $L^2(\mathbb{R}_+, d\mu_\alpha)$, $\mathbb{R}_+ = (0,\infty)$, $d\mu_\alpha(x) = x^\alpha e^{-x}\,dx$
- factorization: $L_\alpha f(x) = -(x^\alpha e^{-x})^{-1}\frac{d}{dx}\big(x^{\alpha+1}e^{-x}\frac{d}{dx}f(x)\big)$

Classical setting: $\alpha > -1$

- Laguerre polynomials L_n^α, $n \ge 0$, form an OgB in $L^2(d\mu_\alpha)$ and are eigenfunctions of L_α, $L_\alpha L_n^\alpha = nL_n^\alpha$
- emerging self-adjoint extension in $L^2(d\mu_\alpha)$

$$L_\alpha^{\mathrm{cls}} f = \sum_{n=0}^{\infty} n\langle f, \breve{L}_n^\alpha\rangle_{d\mu_\alpha}\, \breve{L}_n^\alpha$$

$$\mathrm{Dom}\, L_\alpha^{\mathrm{cls}} = \big\{f \in L^2(d\mu_\alpha) : \Sigma \text{ converges in } L^2(d\mu_\alpha)\big\}$$

Question: what about $\alpha \le -1$?

Differential perspective

Define

$$\mathcal{D}_\alpha = \big\{ f \in L^2(d\mu_\alpha) : L_\alpha f \in L^2(d\mu_\alpha) \big\}$$

The following characterization holds:

- for $\boldsymbol{\alpha \geq 1}$

$$\text{Dom}\, L_\alpha^{\text{cls}} = \mathcal{D}_\alpha \quad \text{and} \quad L_\alpha^{\text{cls}} = L_\alpha$$

- for $\boldsymbol{-1 < \alpha < 1}$

$$\text{Dom}\, L_\alpha^{\text{cls}} = \Big\{ f \in \mathcal{D}_\alpha : \lim_{x \to 0^+} x^{\alpha+1} f'(x) = 0 \Big\} \quad \text{and} \quad L_\alpha^{\text{cls}} = L_\alpha$$

(this in fact for all $\boldsymbol{\alpha > -1}$)

Exotic Laguerre setting (1-dim)

For $\alpha \leq -1$ consider

$$L_\alpha^{\text{exo}} = L_\alpha \ \text{ on } \ \text{Dom}\, L_\alpha^{\text{exo}} = \mathcal{D}_\alpha$$

It turns out that [*Hajmirzaahmad, 1995*]:

- L_α^{exo} is self-adjoint in $L^2(d\mu_\alpha)$, with discrete spectral decomposition given by means of the system

$$\{x^{-\alpha} L_n^{-\alpha} : n \geq 0\} \qquad \text{OgB in } L^2(d\mu_\alpha)$$

- one has $\quad L_\alpha(x^{-\alpha} L_n^{-\alpha}) = (n-\alpha)\, x^{-\alpha} L_n^{-\alpha} \quad$ and

$$L_\alpha^{\text{exo}} f = \sum_{n=0}^{\infty} (n-\alpha) \langle f, x^{-\alpha} \breve{L}_n^{-\alpha} \rangle_{d\mu_\alpha} x^{-\alpha} \breve{L}_n^{-\alpha},$$

$$\text{Dom}\, L_\alpha^{\text{exo}} = \{f \in L^2(d\mu_\alpha) : \Sigma \text{ converges in } L^2(d\mu_\alpha)\}$$

- the latter defines the self-adjoint operator for $\alpha < 1$ and

$$\text{Dom}\, L_\alpha^{\text{exo}} = \Big\{f \in \mathcal{D}_\alpha : \lim_{x \to 0^+} [x f'(x) + \alpha f(x)] = 0\Big\} \ \text{ and } \ L_\alpha^{\text{exo}} = L_\alpha$$

Laguerre semigroups (1-dim)

$\alpha > -1$: classical Laguerre semigroup

$$T_t^\alpha f(x) = \exp\big(-tL_\alpha^{\mathrm{cls}}\big) f(x) = \int_{\mathbb{R}_+} G_t^\alpha(x,y) f(y)\, d\mu_\alpha(y)$$

$$G_t^\alpha(x,y) = \frac{e^{t(\alpha+1)/2}}{2\sinh(t/2)} \exp\Big(-\frac{e^{-t/2}}{2\sinh(t/2)}(x+y)\Big)(xy)^{-\alpha/2} I_\alpha\Big(\frac{\sqrt{xy}}{\sinh(t/2)}\Big)$$

$0 \neq \alpha < 1$: exotic Laguerre semigroup

$$\widetilde{T}_t^\alpha f(x) = \exp\big(-tL_\alpha^{\mathrm{exo}}\big) f(x) = \int_{\mathbb{R}_+} \widetilde{G}_t^\alpha(x,y) f(y)\, d\mu_\alpha(y)$$

$$\widetilde{G}_t^\alpha(x,y) = e^{t\alpha}(xy)^{-\alpha} G_t^{-\alpha}(x,y)$$

for exotic $0 < \alpha < 1$: *pencil phenomenon* restriction

$$\alpha + 1 < p < (\alpha+1)/\alpha$$

General multi-dimensional situation

Now: $d \geq 1$, $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1, \ldots, d\}$

$$A(\mathcal{E}) = \left\{\alpha \in \mathbb{R}^d : 0 \neq \alpha_i < 1 \text{ for } i \in \mathcal{E} \text{ and } \alpha_i > -1 \text{ for } i \in \mathcal{E}^c\right\}$$

$$m_{\mathcal{E}}(\alpha) = \max\{\alpha_i : i \in \mathcal{E}\} \qquad (m_{\emptyset}(\alpha) = -\infty)$$

Define

$$\mathfrak{L}_n^{\alpha,\mathcal{E}} = \bigotimes_{i=1}^{d} \begin{cases} x_i^{-\alpha_i} \breve{L}_{n_i}^{-\alpha_i}, & i \in \mathcal{E}, \\ \breve{L}_{n_i}^{\alpha_i}, & i \notin \mathcal{E} \end{cases}$$

$\{\mathfrak{L}_n^{\alpha,\mathcal{E}} : n \in \mathbb{N}^d\}$ is an OnB in $L^2(\mathbb{R}_+^d, d\mu_\alpha)$, where $\mu_\alpha = \otimes_{i=1}^d \mu_{\alpha_i}$

$$\mathbb{L}_\alpha = \sum_{i=1}^{d} L_{\alpha_i}, \quad \mathbb{L}_\alpha \mathfrak{L}_n^{\alpha,\mathcal{E}} = \lambda_n^{\alpha,\mathcal{E}} \mathfrak{L}_n^{\alpha,\mathcal{E}}, \quad \lambda_n^{\alpha,\mathcal{E}} = \sum_{i=1}^{d} n_i - \sum_{i \in \mathcal{E}} \alpha_i$$

Natural self-adjoint extension in $L^2(d\mu_\alpha)$

$$\mathbb{L}_{\alpha,\mathcal{E}} f = \sum_{n \in \mathbb{N}^d} \lambda_n^{\alpha,\mathcal{E}} \langle f, \mathfrak{L}_n^{\alpha,\mathcal{E}} \rangle_{d\mu_\alpha} \mathfrak{L}_n^{\alpha,\mathcal{E}}, \quad \operatorname{Dom} \mathbb{L}_{\alpha,\mathcal{E}} = \ldots$$

General Laguerre semigroup maximal operator

$\boldsymbol{\alpha \in A(\mathcal{E})}$: general multi-dim Laguerre semigroup

$$\mathbb{T}_t^{\alpha,\mathcal{E}} f(x) = \exp\big(-t\mathbb{L}_{\alpha,\mathcal{E}}\big) f(x) = \int_{\mathbb{R}_+^d} \mathbb{G}_t^{\alpha,\mathcal{E}}(x,y) f(y)\, d\mu_\alpha(y)$$

$$\mathbb{G}_t^{\alpha,\mathcal{E}}(x,y) = \prod_{i\in\mathcal{E}} \widetilde{G}_t^{\alpha_i}(x_i,y_i) \prod_{i\in\mathcal{E}^c} G_t^{\alpha_i}(x_i,y_i)$$

when $\boldsymbol{m_{\mathcal{E}}(\alpha) > 0}$ a pencil phenomenon occurs:

$$1 + m_{\mathcal{E}}(\alpha) < p < 1 + \frac{1}{m_{\mathcal{E}}(\alpha)}$$

Consider the maximal operator

$$\mathbb{T}_*^{\alpha,\mathcal{E}} f = \sup_{t>0} \left| \mathbb{T}_t^{\alpha,\mathcal{E}} \right|$$

Maximal theorem

By Stein's maximal theorem for semigroups [*Topics...*]:
if $\boldsymbol{m_{\mathcal{E}}(\alpha) < 0}$ then $\mathbb{T}_*^{\alpha,\mathcal{E}}$ is bounded on $L^p(d\mu_\alpha)$, $p > 1$

Question: what about weak type $(1,1)$?

Theorem (A.N., P. Sjögren, T.Z. Szarek)

Let $d \geq 1$ and $\alpha \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1, \ldots, d\}$.
Assume that $m_{\mathcal{E}}(\alpha) < 0$. Then

$$\mu_\alpha\big\{x \in \mathbb{R}^d_+ : \mathbb{T}_*^{\alpha,\mathcal{E}} f(x) > \lambda\big\} \leq \frac{C}{\lambda} \int_{\mathbb{R}^d_+} |f(x)|\, d\mu_\alpha(x)$$

uniformly in $\lambda > 0$ and $f \in L^1(d\mu_\alpha)$, with C indep. of λ and f.

Remark. Classical $\mathcal{E} = \emptyset$:
[*Muckenhoupt 1969*] (one-dim), [*Dinger 1992*], [*Sasso 2005*]

Other classical settings

Exotic counterparts exist in many other classical situations, e.g.

- Bessel (related to the Hankel transform)

$$B_\nu = -\frac{d^2}{dx^2} - \frac{2\nu+1}{x}\frac{d}{dx} \qquad \text{on } \mathbb{R}_+$$

- Jacobi (related to Jacobi polynomials)

$$J_{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - \left[\beta-\alpha-(\alpha+\beta+2)x\right]\frac{d}{dx} \qquad \text{on } (-1,1)$$

Maximal operators of exotic and non-exotic Laguerre and other semigroups associated with classical orthogonal expansions, by A.N., P. Sjögren and T.Z. Szarek, see `arXiv`

Bessel setting (1-dim)

Bessel 'Laplacian'

$$B_\nu = -\frac{d^2}{dx^2} - \frac{2\nu+1}{x}\frac{d}{dx}$$

- symmetric in $L^2(\mathbb{R}_+, d\eta_\nu)$, $d\eta_\nu(x) = x^{2\nu+1}\,dx$

Classical setting: $\nu > -1$

- let $\varphi_z^\nu(x) = (xz)^{-\nu} J_\nu(xz)$ for $x, z > 0$, then $B_\nu \varphi_z^\nu = z^2 \varphi_z^\nu$
- (modified) **Hankel transform** is given by

$$h_\nu f(z) = \int_0^\infty f(x)\varphi_z^\nu(x)\,d\eta_\nu(x), \qquad z > 0$$

  h_ν extends to an isometry on $L^2(d\eta_\nu)$ and $h_\nu^{-1} = h_\nu$
- emerging self-adjoint extension in $L^2(d\eta_\nu)$

$$B_\nu^{\mathrm{cls}} f = h_\nu\big(z^2 h_\nu f(z)\big)$$

$$\mathrm{Dom}\, B_\nu^{\mathrm{cls}} = \big\{ f \in L^2(d\eta_\nu) : z^2 h_\nu f(z) \in L^2(d\eta_\nu) \big\}$$

Exotic Bessel setting (1-dim)

Exotic setting: let now $\mathbf{0 \neq \nu < 1}$

- observe $x \mapsto (xz)^{-2\nu}\varphi_z^{-\nu}(x)$ is an eigenfunction of B_ν with the eigenvalue z^2
- introduce **exotic** (modified) **Hankel transform**

$$\widetilde{h}_\nu f(z) = \int_0^\infty f(x)(xz)^{-2\nu}\varphi_z^{-\nu}(x)\, d\eta_\nu(x), \qquad z > 0$$

$\widetilde{h}_\nu$ extends to an isometry on $L^2(d\eta_\nu)$ and $\widetilde{h}_\nu^{-1} = \widetilde{h}_\nu$

- can define self-adjoint extension of B_ν in $L^2(d\eta_\nu)$ as

$$B_\nu^{\mathrm{exo}} f = \widetilde{h}_\nu\big(z^2 \widetilde{h}_\nu f(z)\big)$$
$$\mathrm{Dom}\, B_\nu^{\mathrm{exo}} = \big\{ f \in L^2(d\eta_\nu) : z^2 \widetilde{h}_\nu f(z) \in L^2(d\eta_\nu)\big\}$$

Bessel semigroups (1-dim)

$\boldsymbol{\nu > -1}$: classical $\exp\big(-tB_\nu^{\mathrm{cls}}\big)$ with the kernel

$$W_t^\nu(x,y) = \int_0^\infty e^{-tz^2}\varphi_z^\nu(x)\varphi_z^\nu(y)\,d\eta_\nu(z)$$
$$= \frac{1}{2t}\exp\Big(-\frac{1}{4t}(x^2+y^2)\Big)(xy)^{-\nu}I_\nu\Big(\frac{xy}{2t}\Big)$$

$\boldsymbol{0 \neq \alpha < 1}$: exotic $\exp\big(-tB_\nu^{\mathrm{exo}}\big)$ with the kernel

$$\widetilde{W}_t^\nu(x,y) = (xy)^{-2\nu}W_t^{-\nu}(x,y)$$

Probabilistic background

- classical Bessel semigroup is Markovian for all $\nu > -1$
- exotic Bessel semigroup is (strictly) submarkovian for $\nu < 0$

These are transition semigroups for Bessel processes

General multi-dimensional situation

With $\nu \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1, \ldots, d\}$ can define self-adjoint in $L^2(d\eta_\nu)$ operator $\mathbb{B}_{\nu,\mathcal{E}}$ and consider the semigroup

$$\mathbb{W}_t^{\nu,\mathcal{E}} = \exp(-t\mathbb{B}_{\nu,\mathcal{E}})$$

The integral kernel of this semigroup is

$$\mathbb{W}_t^{\nu,\mathcal{E}}(x,y) = \prod_{i\in\mathcal{E}} \widetilde{W}_t^{\nu_i}(x_i,y_i) \prod_{i\in\mathcal{E}^c} W_t^{\nu_i}(x_i,y_i)$$

For the maximal operator

$$\mathbb{W}_*^{\nu,\mathcal{E}} f = \sup_{t>0} \left|\mathbb{W}_t^{\nu,\mathcal{E}} f\right|$$

we obtain

Theorem (A.N., P. Sjögren, T.Z. Szarek)

Let $d \geq 1$ and $\nu \in A(\mathcal{E})$ for some $\mathcal{E} \subset \{1, \ldots, d\}$.
Assume that $m_{\mathcal{E}}(\nu) < 0$. Then $\mathbb{W}_*^{\nu,\mathcal{E}}$ is weak type $(1,1)$.

Remark. Classical $\mathcal{E} = \emptyset$ ...

Thank you for your attention