Operators related to the Laplacian with drift in Euclidean space

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Joint work with Hong-Quan Li (Shanghai)

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$$\Delta_v = \Delta + 2v \cdot \nabla \quad = \sum_{i=1}^{n} \Big(\frac{\partial^2}{\partial x_i^2} + 2v_i \frac{\partial}{\partial x_i} \Big),$$

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$$\mu\{x; |R_D f(x)| > \lambda\} \leq C \int \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)^{\frac{q}{2}-1} d\mu, \qquad \lambda > 0,$$

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This inequality is sharp in the sense that q cannot be replaced by any smaller number.

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For the heat semigroup, the vertical Littlewood-Paley-Stein function of order k is usually defined as

$$\mathcal{H}_k(f)(x) = \left(\int_0^{+\infty} \left|t^{\frac{k}{2}}\nabla^k e^{t\Delta_v} f(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}},$$

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The operator h_k is of weak type $(1,1)$ if and only if $k \leq 1$. When $k > 1$, h_k is bounded from $L(1+\ln^+ L)^{\frac{k}{2}-\frac{3}{4}}(\mu)$ into $L^{1,\infty}(\mu)$, and this is optimal.

The "maximal operator" H_k is a bit better:

Theorem

The operator H_k is of weak type $(1,1)$ if and only if $k \leq 2$. When $k > 2$, H_k is bounded from $L(1+\ln^+ L)^{\frac{k}{2}-1}(\mu)$ into $L^{1,\infty}(\mu)$, and this is optimal.

The analogous operators defined with the Poisson semigroup are of weak type $(1,1)$ for all values of k.

Sketches of proofs

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Proposition

For $|x-y|>1$

$$|R_D(x,y)| \lesssim e^{-x_1-y_1-|x-y|}\,|x-y|^{\frac{q-n-1}{2}}\left[1+\left(\frac{|x'-y'|^2}{|x-y|}\right)^{\frac{k}{2}}\right].$$

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From this proposition, it can be proved that R_D is of weak type $(1,1)$ for $q=1$ and 2.

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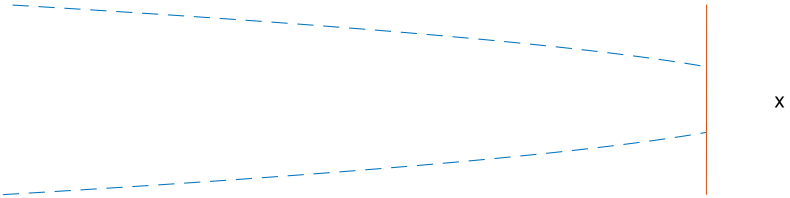
Here the main part is given by the parabolic region $|x' - y'| < \sqrt{x_1 - y_1}$.

this is the y plane



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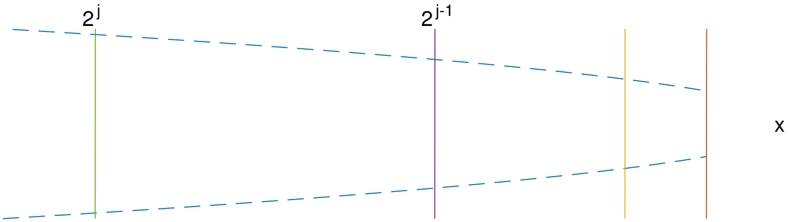

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where ψ_j is the normalized dilation of the function

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Putting things together, we see that R_D is controlled by the sum over $j = 1, 2, \ldots$ of the operators

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Notice that

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By summing in j, we get the weak type (1,1) of R_D, for $q = 1$. $\square$

Some hints for the case $q = 2$

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We now have the kernel

$$e^{-2x_1}(x_1 - y_1)^{\frac{1-n}{2}} \exp\left(-\frac{1}{4}\,\frac{\frac{|x'-y'|^2}{x_1-y_1}}{1+\frac{|x'-y'|}{\sqrt{x_1-y_1}}}\right)\chi_{\{x_1-y_1>1\}},$$

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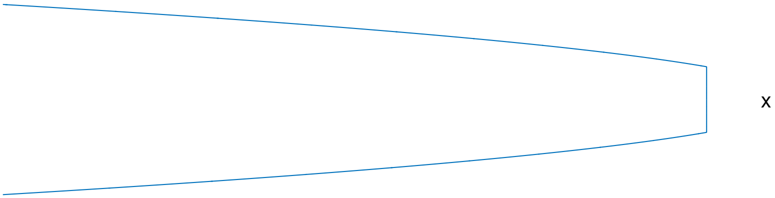
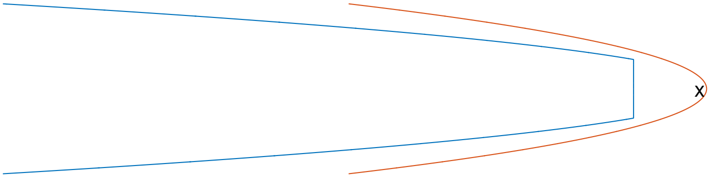
$$Sg(x) = e^{-2x_1} \int_{y_1<x_1-1} (x_1 - y_1)^{\frac{1-n}{2}} \int_{|x'-y'|<\sqrt{x_1-y_1}} g(y_1, y')\, dy'\, dy_1 .$$

It must be proved that S is bounded from $L^1(dy)$ into $L^1(\mu)$.


x


x

THE END

THANK YOU