On potential theory of subordinate killed processes

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Table of Contents

Table of Contents

Joint work with P. Kim (SNU) and R. Song (UIUC)

Killed stable process

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For $D\subset\mathbb{R}^d$ open, let $\tau_D:=\inf\{t>0:\ Z_t\notin D\}$, $Z_t^D:=Z_t$ if $t<\tau_D$, ∂ (cemetary) otherwise, $Q_t^Df(x):=\mathbb{E}_xf(Z_t^D)=\mathbb{E}_x(f(Z_t),t<\tau_D)$ the corresponding semigroup.

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$$\mathcal{L}_1 f := \lim_{t\to 0} \frac{Q_t^D f - f}{t}$$

a possible definition of fractional Laplacian in D; usually called *fractional Laplacian in D with zero exterior condition:*. Notation: $-(-\Delta)^{\alpha/2}_{|D}$.

KSBM and SKBM

Let $W = (W_t, \mathbb{P}_x)$ be a Brownian motion in $\mathbb{R}^d$, $S = (S_t)_{t \geq 0}$ an independent $\alpha/2$-stable subordinator. Then W_{S_t} is a subordinate Brownian motion and $(Z_t) \stackrel{d}{=} (W_{S_t})$.

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W^D Brownian motion killed upon exiting D, $Y_t^D := W_{S_t}^D$ is a subordinate killed Brownian motion (SKBM). If $(P_t^D)_{t\geq 0}$ is the semigroup of W^D, then the infinitesimal generator of Y^D is

$$\mathcal{L}_0 f = -(-\Delta_{|D})^{\alpha/2} f := \frac{1}{|\Gamma(-\alpha/2)|} \int_0^\infty (P_t^D f - f) t^{-\alpha/2-1} dt$$

Another possible definition of a fractional Laplacian in D: *the fractional power of Dirichlet Laplacian*.

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Another possible definition of a fractional Laplacian in D: *the fractional power of Dirichlet Laplacian*. $\mathcal{L}_0 \neq \mathcal{L}_1$

If $(\widetilde{Q}_t^D)_{t\geq 0}$ is the semigroup of Y^D, then $\widetilde{Q}_t^D f \leq Q_t^D f$, $f \geq 0$. Y^D is a "smaller" process then Z^D.

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Let S be a $\beta/2$-stable subordinator, $\beta \in (0,2]$, T a $\gamma/2$-stable subordinator, $\gamma \in (0,2)$, so that $(\beta/2)(\gamma/2) = \alpha/2$. Let $Z_t = W_{S_t}$ be a SBM, Z^D the KSBM, and $Y_t^D = Z_{T_t}^D$ the subordinate killed Z.

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$$\mathcal{L} = -((-\Delta)_{|D}^{\beta/2})^{\gamma/2}\,.$$

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Goal: study potential theory of operators like $\mathcal{L}$ (i.e. potential theory of Y^D) and see how it depends on β and γ.

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Various processes:

$Z_t := W_{S_t}$ subordinate Brownian motion

$X_t := Z_{T_t} = W_{(T\circ S)_t}$ subordinate Z, twice SBM

Z^D killed Z, KSBM

X^D killed X, KSBM

$Y_t^D := Z^D_{T_t}$ subordinate Z^D, subordinate KSBM

Assumptions

Both ϕ and ψ are complete Bernstein functions (unless $\phi(\lambda) = \lambda$),

$$\psi(\lambda) = \int_0^\infty (1 - e^{-\lambda t})\nu(t)\, dt\,, \quad \nu \text{ completely monotone.}$$

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Weak scaling conditions at infinity: $0<\delta_1\le\delta_2<1$, $0<\gamma_1\le\gamma_2<1$:

$$a_1\left(\frac{R}{r}\right)^{\delta_1} \le \frac{\phi(R)}{\phi(r)} \le a_2\left(\frac{R}{r}\right)^{\delta_2}, \quad 1<r\le R<\infty\,,$$

$$b_1\left(\frac{R}{r}\right)^{\gamma_1} \le \frac{\psi(R)}{\psi(r)} \le b_2\left(\frac{R}{r}\right)^{\gamma_2}, \quad 1<r\le R<\infty\,.$$

Consequence: scaling for $\psi\circ\phi$ with exponents $0<\gamma_1\delta_1\le\gamma_2\delta_2<1$.

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Consequence: scaling for $\psi\circ\phi$ with exponents $0<\gamma_1\delta_1\le\gamma_2\delta_2<1$.
If $\phi(\lambda)=\lambda$, weaker conditions on ψ (include non-power behavior).

Transition densities of Z: $p(t,x,y)=p(t,|x-y|)$ where

$$p(t,r)=(4\pi t)^{-d/2}e^{-r^2/4t}\,,\quad S_t=t\,,$$
$$p(t,r)=\int_0^\infty (4\pi s)^{-d/2}e^{-r^2/4s}\mathbb{P}(S_t\in ds)\,,\quad \text{otherwise.}$$

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Transition densities of Z^D:

$$p^D(t,x,y)=p(t,x,y)-\mathbb{E}_x[p(t-\tau_D,Z_{\tau_D},y),\tau_D<t]\,,\quad t>0,\ x,y\in D\,.$$

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$$p^D(t,x,y) = p(t,x,y) - \mathbb{E}_x[p(t-\tau_D, Z_{\tau_D}, y), \tau_D < t]\,, \quad t>0,\ x,y \in D\,.$$

Sharp two-sided estimates: (i) $\phi(\lambda) = \lambda$; for $t \le T$, $x,y \in D$,

$$p^D(t,x,y) \asymp \mathbb{P}_x(t < \tau_D^W)\,\mathbb{P}_y(t<\tau_D^W) t^{-d/2} e^{-\frac{c|x-y|^2}{t}}\,,$$
$$p^D(t,x,y) \asymp \left(\frac{\delta_D(x)}{t^{1/2}} \wedge 1\right)\left(\frac{\delta_D(y)}{t^{1/2}} \wedge 1\right) t^{-d/2} e^{-\frac{c|x-y|^2}{t}}$$

Varopoulos 2003 (D Lipschitz), Zhang 2003, Song 2004 (D $C^{1,1}$)

(ii) ϕ satisfies weak scaling at infinity, $\Phi(r) = 1/\phi(r^{-2})$: $t \le T$, $x, y \in D$, D is κ-fat open set,

$$p^D(t,x,y) \asymp \mathbb{P}_x(t < \tau_t^Z)\mathbb{P}_y(t < \tau_D^Z)\left(\Phi^{-1}(t)^{-d} \wedge \frac{t}{|x-y|^d\Phi(|x-y|)}\right).$$

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D is $C^{1,1}$ open set, $x,y\in D$,

$$\begin{aligned} p^D(t,x,y) &\asymp \left(\frac{\Phi(\delta_D(x))^{1/2}}{t^{1/2}}\wedge 1\right)\left(\frac{\Phi(\delta_D(y))^{1/2}}{t^{1/2}}\wedge 1\right) \\ &\quad\times\left(\Phi^{-1}(t)^{-d}\wedge \frac{t}{|x-y|^d\Phi(|x-y|)}\right), \quad t\le T, \\ p^D(t,x,y) &\asymp e^{-\lambda_1 t}\Phi(\delta_D(x))^{1/2}\Phi(\delta_D(y))^{1/2}, \quad t\ge T. \end{aligned}$$

Chen, Kim, Song 2014; Bogdan, Grzywny, Ryznar 2014

Green function and jumping kernel

Green functions of Y^D and X: For $x, y \in D$,

$$G^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)v(t)\,dt \le \int_0^\infty p(t,x,y)v(t)\,dt = G^X(x,y)\,.$$

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Here v is the potential density of (T_t) and under weak scaling $v(t) \asymp \frac{1}{t\psi(t^{-1})}$, $t \le 1$.

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Jumping kernels of of Y^D and X: For $x, y \in D$,

$$J^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)\nu(t)\,dt \le \int_0^\infty p(t,x,y)\nu(t)\,dt = J^X(x,y)\,.$$

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Here ν is the Lévy density of (T_t) and under weak scaling $\nu(t) \asymp \frac{\psi(t^{-1})}{t}$, $t \le 1$.

κ-fat and $C^{1,1}$-open sets

Let $0 < \kappa < 1$. An open set $D \subset \mathbb{R}^d$ is said to be κ-fat if there is $R_1 > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, R_1]$, there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$. The pair (R_1, κ) is called the characteristics of the κ-fat open set D.

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Let $U \subset \mathbb{R}^d$ be an open set and let $Q \in \partial U$. We say that U is $C^{1,1}$ near Q if there exist a localization radius $R > 0$, a $C^{1,1}$-function $\varphi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi_Q(0) = 0$, $\nabla\varphi_Q(0) = (0, \dots, 0)$, $\|\nabla\varphi_Q\|_\infty \le \Lambda$, $|\nabla\varphi_Q(z) - \nabla\varphi_Q(w)| \le \Lambda|z - w|$, and an orthonormal coordinate system CS_Q with its origin at Q such that $B(Q, R) \cap U = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \varphi_Q(\widetilde{y})\}$ where $\widetilde{y} := (y_1, \dots, y_{d-1})$. The pair (R, Λ) is called the $C^{1,1}$ characteristics of U near Q. An open set $U \subset \mathbb{R}^d$ is said to be a (uniform) $C^{1,1}$ open set with characteristics (R, Λ) if it is $C^{1,1}$ with characteristics (R, Λ) near every boundary point $Q \in \partial U$.

Harmonic functions

$(X_t, \mathbb{P}_x)$ a strong Markov process in a metric space $\mathfrak{X}$. A non-negative $u : \mathfrak{X} \to [0, \infty)$ is harmonic in an open $U \subset \mathfrak{X}$ (wrt the process X) if

$$u(x) = \mathbb{E}_x \left(u(X_{\tau_B}) \right) , \quad \text{for all bounded open } B \subset U \text{ and for all } x \in B.$$

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By the strong Markov property, regular harmonic implies harmonic.

Green functions on small open sets

Proposition: Let $D \subset \mathbb{R}^d$ be bounded, $R > 0$ and $\Lambda > 0$. There exist $b > 2$ and $C = C(R, \Lambda, \phi, \psi, d) \in (0,1)$ such that for every $r \in (0, 1/b]$ and every $C^{1,1}$ open set $U \subset D$ with characteristics $(rR, \Lambda/r)$ and $\mathrm{diam}(U) \le r$ satisfying $\mathrm{dist}(U, \partial D) \ge (b+2)r$, we have

$$C G_U^X(x,y) \le G_U^{Y^D}(x,y) \le G_U^X(x,y)\,, \quad x, y \in U\,.$$

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Define $F(x,y) := \frac{J^{Y^D}(x,y)}{J^X(x,y)} - 1 \in (-1, 0]$.

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$$CG_U^X(x, y) \le G_U^{Y^D}(x, y) \le G_U^X(x, y), \quad x, y \in U.$$

Define $F(x, y) := \frac{J^{Y^D}(x,y)}{J^X(x,y)} - 1 \in (-1, 0]$. Then, for any open $U \subset D$ with $\mathrm{diam}(U) < r$ and $\mathrm{dist}(U, \partial D) \ge br$ ($b > 2$ not depending on U and r), it holds that

$$|F(x, y)| < \frac{1}{2}, \quad x, y \in U.$$

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The Dirichlet form corresponding to the perturbed semigroup $(T_t^U)_{t\ge 0}$ was computed by Chen, Song 2003:

$$\mathcal{Q}(f,f) = \mathcal{E}^{X^U}(f,f) - \int_U \int_U f(x)f(y)F(x,y)J^X(x,y)\,dy\,dx$$

with $\mathcal{D}(\mathcal{E}^{X^U})$ as the domain.
We show that $(\mathcal{Q}, \mathcal{D}(\mathcal{E}^{X^U})) = (\mathcal{E}^{Y^{D,U}}, \mathcal{D}(\mathcal{E}^{Y^{D,U}}))$ is the Dirichlet form of Y^D killed upon exiting U.

Consequently, if V^U denotes the Green function of the semigroup $(T_t^U)_{t\geq 0}$, then $V^U = G_U^{Y^D}$ - the Green function of Y^D killed upon exiting U.

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On the other hand,

$$V^U(x,y) = u^U(x,y) G_U^X(x,y), \quad x,y \in U,$$

where

$$u^U(x,y) := \mathbb{E}_x^y[K_{\tau_X^U}^U] \leq 1$$

is the conditional gauge for (K_t^U).

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$$u^U(x,y) \geq c, \quad x,y \in U\,.$$

Consequently, if V^U denotes the Green function of the semigroup $(T_t^U)_{t\geq 0}$, then $V^U = G_U^{Y^D}$ - the Green function of Y^D killed upon exiting U.

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$$u^U(x,y) \geq c, \quad x,y \in U\,.$$

With this we have that

$$G_U^{Y^D}(x,y) \asymp G_U^X(x,y), \quad x,y \in U.$$

Harnack inequalities

Theorem: Suppose that $D \subset \mathbb{R}^d$ is a bounded κ-fat (Lipschitz when $\phi(\lambda) = \lambda$) open set. There exists a constant $C > 0$ such that for any $r \in (0, 1]$ and $B(x_0, r) \subset D$ and any Borel function f which is non-negative in D and harmonic in $B(x_0, r)$ with respect to Y^D, we have

$$f(x) \le Cf(y), \qquad \text{for all } x, y \in B(x_0, r/2).$$

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$$f(x) \leq Cf(y), \qquad \text{for all } x, y \in B(x_0, r/2).$$

Theorem: Suppose that $D \subset \mathbb{R}^d$ is a bounded κ-fat open set. There exists a constant $C = C(\phi, \psi, \mathrm{diam}(D)) > 1$ such that the following is true: If $L > 0$ and $x_1, x_2 \in D$ and $r \in (0, 1)$ are such that $|x_1 - x_2| < Lr$ and $B(x_1, r) \cup B(x_2, r) \subset D$, then for any Borel function f which is non-negative in D and harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to Y^D, we have

$$C^{-1}(L \vee 1)^{-d-\delta_2} f(x_2) \leq f(x_1) \leq C(L \vee 1)^{d+\delta_2} f(x_2).$$

One of the main ingredients of the proof is the following comparison of jumping kernel.

Lemma: Suppose that $D \subset \mathbb{R}^d$ is a bounded κ-fat (Lipschitz when $\phi(\lambda) = \lambda$) open set. For every $\varepsilon_0 \in (0, 1]$, there exists a constant $C \geq 1$ such that for all $x_0 \in D$ and all $r \leq 1$ satisfying $B(x_0, (1 + \varepsilon_0)r) \subset D$, it holds that

$$J^{Y^D}(z, x_1) \leq C J^{Y^D}(z, x_2), \quad x_1, x_2 \in B(x_0, r), \quad z \in D \setminus B(x_0, (1 + \varepsilon_0)r).$$

Boundary Harnack principle in the interior of D

Theorem: Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded κ-fat (Lipschitz when $\phi(\lambda) = \lambda$) open set. There exists a constant $b = b(\phi, \psi, d) > 2$ such that, for every open set $E \subset D$ and every $Q \in \partial E \cap D$ such that E is $C^{1,1}$ near Q with characteristics $(\delta_D(Q) \wedge 1, \Lambda)$, the following holds: There exists a constant $C = C(\delta_D(Q) \wedge 1, \Lambda, \psi, \phi, d) > 0$ such that for every $r \leq (\delta_D(Q) \wedge 1)/(b+2)$ and every non-negative function f on D which is regular harmonic in $E \cap B(Q, r)$ with respect to Y^D and vanishes on $E^c \cap B(Q, r)$, we have for $x, y \in E \cap B(Q, 2^{-6}\kappa_0^4 r)$,

$$\sqrt{(\psi \circ \phi)(\delta_E(x)^{-2})} f(x) \leq C \sqrt{(\psi \circ \phi)(\delta_E(y)^{-2})} f(y) \,,$$

where $\kappa_0 = (1 + (1 + \Lambda)^2)^{-1/2}$.

Green function estimates in $C^{1,1}$-open set

From now on, D is a bounded $C^{1,1}$-open set in $\mathbb{R}^d$, $d \geq 3$.

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Note that $g(|x-y|)$ is the Green function estimate for X when $|x-y|$ is bounded.

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Theorem: There exists a constant $C \geq 1$ such that for all $x, y \in D$,

$$G^{Y^D}(x,y) \asymp^C \left(\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(|x-y|)^{1/2}} \wedge 1 \right) \left(\frac{\Phi(\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}} \wedge 1 \right) g(|x-y|).$$

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The boundary behavior $\Phi(\delta_D(x))^{1/2}$ completely determined by ϕ; same as the one of Z^D.

Outline of the proof: With $T = 2\Phi(\mathrm{diam}(D))$, write

$$G^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)v(t)dt = \int_0^{\Phi(|x-y|)} + \int_{\Phi(|x-y|)}^T + \int_T^\infty,$$

use sharp estimates of $p^D(t,x,y)$ and $v(t)$, and the fact that v is decreasing.

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use sharp estimates of $p^D(t,x,y)$ and $v(t)$, and the fact that v is decreasing.

For the upper bound estimate all three integrals, for the lower only the first which is the dominating term.

Jumping kernel estimates in $C^{1,1}$-open sets

Let

$$j(r) := \frac{\psi(\Phi(r)^{-1})}{r^d}, \quad r > 0.$$

Note that $j(|x-y|)$ is the Lévy density estimate for X when $|x-y|$ is bounded.

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Note that $j(|x-y|)$ is the Lévy density estimate for X when $|x-y|$ is bounded.

Theorem: Suppose $\phi(\lambda) = \lambda$. There exists a constant $C \geq 1$ such that for all $x, y \in D$,

$$J^{Y^D}(x,y) \asymp^C \left(\frac{\delta_D(x)}{|x-y|} \wedge 1\right) \left(\frac{\delta_D(y)}{|x-y|} \wedge 1\right) j(|x-y|).$$

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Boundary behavior same as for the Green function. The proof similar, uses $J^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)\nu(t)dt$, and estimate of ν.

Jumping kernel estimates in $C^{1,1}$-open sets (cont.)

Case $\phi(\lambda) \neq \lambda$. Set

$$\theta(t) := \Phi(t)\psi(\Phi(t)^{-1}) \quad \text{and} \quad \eta(t) := \Phi(t)^{1/2}\psi(\Phi(t)^{-1}), \quad t > 0.$$

Jumping kernel estimates in $C^{1,1}$-open sets (cont.)

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$$\theta(t) := \Phi(t)\psi(\Phi(t)^{-1}) \quad \text{and} \quad \eta(t) := \Phi(t)^{1/2}\psi(\Phi(t)^{-1}), \quad t > 0.$$

Theorem: (i) Suppose that $r \mapsto r^{1/2}\psi(r^{-1})$ is almost decreasing near 0 and for each $T > 0$ there is a constant $C = C(T, \psi) > 0$ such that

$$\int_r^T s^{-1/2}\psi(s^{-1})\, ds \leq C\, r^{1/2}\psi(r^{-1}) \qquad \text{for } r \in (0, T].$$

Then there exists $c \geq 1$ such that for all $x, y \in D$,

$$J^{Y^D}(x,y) \asymp^c \left(\frac{\theta(\delta_D(x) \wedge \delta_D(y))}{\theta(|x-y|)} \wedge 1\right) j(|x-y|).$$

Jumping kernel estimates in $C^{1,1}$-open sets (cont.)

(ii) Suppose that $r^{1/2}\psi(r^{-1})$ is almost increasing near 0 and for every $T > 0$, there is a constant $C = C(T, \psi) > 0$ so that

$$\int_0^r s^{-1/2}\psi(s^{-1})ds \le Cr^{1/2}\psi(r^{-1}) \qquad \text{for } r \in (0, T].$$

Then there exists $c \ge 1$ such that for all $x, y \in D$,

$$J^{Y^D}(x,y) \asymp^c \left(\frac{\Phi(\delta_D(x) \wedge \delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}} \wedge 1\right)\left(\frac{\eta(\delta_D(x) \vee \delta_D(y))}{\eta(|x-y|)} \wedge 1\right) j(|x-y|)$$

Jumping kernel estimates in $C^{1,1}$-open sets (cont.)

(ii) Suppose that $r^{1/2}\psi(r^{-1})$ is almost increasing near 0 and for every $T>0$, there is a constant $C=C(T,\psi)>0$ so that

$$\int_0^r s^{-1/2}\psi(s^{-1})ds \le Cr^{1/2}\psi(r^{-1}) \qquad \text{for } r\in(0,T].$$

Then there exists $c\ge 1$ such that for all $x,y\in D$,

$$J^{Y^D}(x,y) \asymp^c \left(\frac{\Phi(\delta_D(x)\wedge\delta_D(y))^{1/2}}{\Phi(|x-y|)^{1/2}}\wedge 1\right)\left(\frac{\eta(\delta_D(x)\vee\delta_D(y))}{\eta(|x-y|)}\wedge 1\right)j(|x-y|)$$

Case (i) holds if ψ satisfies weak scaling with exponent $\gamma_1>1/2$.
Case (ii) holds if ψ satisfies weak scaling with exponent $\gamma_2<1/2$.

Example

Let $\psi(\lambda) = \lambda^\gamma$ and assume that $\phi(\lambda) \neq \lambda$. Fix $y \in D$. As $\delta_D(x) \to 0$, we have

$$J^{Y^D}(x,y) \asymp^c \begin{cases} \Phi(\delta_D(x))^{1/2}, & 0 < \gamma < 1/2, \\ \Phi(\delta_D(x))^{1/2} \log(1/\Phi(\delta_D(x))), & \gamma = 1/2, \\ \Phi(\delta_D(x))^{1/2}\, \Phi(\delta_D(x))^{1/2-\gamma}, & 1/2 < \gamma < 1. \end{cases}$$

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Note that the boundary behavior of J^{Y^D} is determined by ϕ, but also by ψ.

In case $\phi(\lambda)=\lambda$, in all three cases

$$J^{Y^D}(x,y)\asymp^c \delta_D(x)\,.$$

The proof of the theorem uses that

$$J^{Y^D}(x,y) = \int_0^\infty p^D(t,x,y)\nu(t)dt\,,$$

and estimates of $p^D(t,x,y)$ and ν. The integral is split into three parts:

$$\int_0^{\Phi(|x-y|)} + \int_{\Phi(|x-y|)}^T + \int_T^\infty\,.$$

The last two integrals are estimated from above in a rather straightforward way, but the first one is quite delicate. Estimates used for the Green function do not work, because some of the integrals diverge.

Theorem: Let $D \subset \mathbb{R}^d$ be a bounded $C^{1,1}$ open set with $C^{1,1}$ characteristics (R, Λ). There exists a constant $C = C(d, \Lambda, R, \phi, \psi) > 0$ such that for any $r \in (0, R]$, $Q \in \partial D$, and any non-negative function f in D which is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$\frac{f(x)}{\Phi(\delta_D(x))^{1/2}} \le C \frac{f(y)}{\Phi(\delta_D(y))^{1/2}} \qquad \text{for all } x, y \in D \cap B(Q, r/2).$$

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As a consequence, the rate of decay of harmonic functions with respect to Y^D is given by $\Phi(\delta_D(x))^{1/2}$, depends on ϕ only, and is equal to the decay of harmonic functions with respect to Z.

Outline of the proof of the BHP

Theorem (Carleson estimate): There exists a constant $C = C(R, \Lambda) > 0$ such that for every $Q \in \partial D$, $0 < r < R/2$, and every non-negative function f in D that is harmonic in $D \cap B(Q, r)$ with respect to Y^D and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$f(x) \leq Cf(x_0) \qquad \text{for } x \in D \cap B(Q, r/2),$$

where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) \geq r/2$.

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where $x_0 \in D \cap B(Q, r)$ with $\delta_D(x_0) \geq r/2$.

In case when $\phi(\lambda) \neq \lambda$, Carleson estimate is proved when D is κ-fat and satisfies the local exterior volume condition. In this case, the proof does not use the explicit estimates for J^{Y^D}.

The proof of the BHP is probabilistic and uses "the box method". Let $x \in B(Q, 2^{-7}\kappa_0 r)$, ($\kappa_0 = (1+(1+\Lambda)^2)^{-1/2}$), $Q_x \in \partial D$ so that $|x - Q_x| = \delta_D(x)$ and CS coordinate system with origin at Q_x such that

$$B(Q_x, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS : y_d > \varphi(\widetilde{y})\}.$$

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Let $x_0 = Q_x + \frac{r}{8}(x - Q_x)/|x - Q_x|$.
For any $a, b > 0$, define "the box"

$$D(a,b) := \left\{ y = (\widetilde{y}, y_d) \text{ in } CS : 0 < y_d - \varphi(\widetilde{y}) < 2^{-2}\kappa_0 ra,\ |\widetilde{y}| < 2^{-2}\kappa_0 rb \right\}.$$

Let $V(1)$ be a $C^{1,1}$ subset of D such that $D(1/2, 1/2) \subset V(1) \subset D(1, 1)$.

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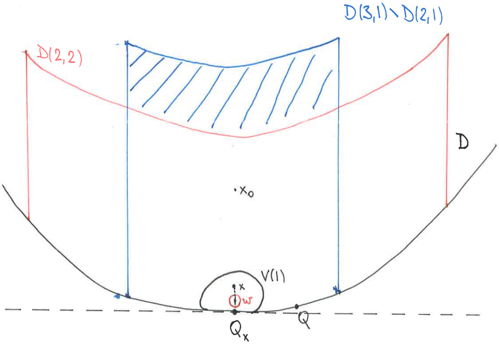
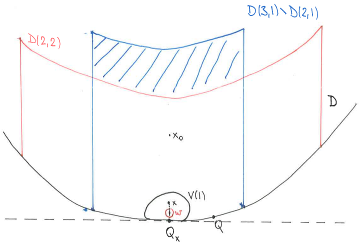


Lower bound:

$$\begin{aligned} f(x) &\geq \mathbb{E}_x\left[f\left(Y^D(\tau_{V(1)})\right); Y^D_{\tau_{V(1)}} \in D(3,1) \setminus D(2,1)\right] \\ &\geq (\mathrm{HI}) \geq c_1 f(x_0)\mathbb{P}_x\left(Y^D(\tau_{V(1)}) \in D(3,1) \setminus D(2,1)\right) \\ &\geq \cdots \geq c_2 f(x_0)\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(r)^{1/2}}. \end{aligned}$$



Upper bound:

$$
\begin{aligned}
f(x) &= \mathbb{E}_x\left[f\left(Y^D(\tau_{V(1)})\right); \; Y^D(\tau_{V(1)}) \in D(2,2)\right] \\
&\quad + \mathbb{E}_x\left[f\left(Y^D(\tau_{V(1)})\right); \; Y^D(\tau_{V(1)}) \notin D(2,2)\right] \\
&= I_1 + I_2 \,.
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$$\begin{aligned} I_1 &\leq (\text{Carleson}) \leq c_3 f(x_0)\mathbb{P}_x(Y^D_{\tau V(1)} \in D(2,2)) \\ &\leq \cdots \leq c_4 f(x_0)\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(r)^{1/2}}. \end{aligned}$$

Upper bound:

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For I_2, let $w = (\widetilde{0}, 2^{-6}\kappa_0 r)$, and show

$$I_2 \le c_5 f(w)\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(r)^{1/2}} \le (\text{Carleson}) \le c_6 f(x_0)\frac{\Phi(\delta_D(x))^{1/2}}{\Phi(r)^{1/2}}.$$

The lower and upper estimate together give that

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Similarly, for any $y \in B(Q, 2^{-6}\kappa_0 r)$,

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By Harnack inequality, $f(x_0) \asymp f(y_0)$ implying

$$\frac{f(x)}{f(y)} \asymp \frac{\Phi(\delta_D(x))^{1/2}}{\Phi(\delta_D(y))^{1/2}}\,.$$