# On purely discontinuous additive functionals of subordinate Brownian motions

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# Introduction

Let $X = (X_t, \mathbb{P}_x)$ be an isotropic Lévy processes in $\mathbb{R}^d$, $d \geq 1$, with Lévy triplet $(0, 0, \nu)$.
For a bounded and symmetric function $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ vanishing on the diagonal, define the purely discontinuous additive functional

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Q: Find sufficient conditions on $F$ such that

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A1: $X$ isotropic $\alpha$-stable process, $\alpha \in (0, 2 \wedge d)$
(Schilling, Vondraček, *Trans.Amer.Math.Soc.*, 2017)

A2: $X$ subordinate Brownian motion with a global scaling property
(Vondraček, Wagner, *Preprint*, 2017)

# Main assumptions (1) and (2)

Let $X = (X_t)_{t\geq 0}$ be a SBM, with the Laplace exponent of the subordinator $\phi$

$$\mathbb{E}[e^{i(\xi, X_t)}] = e^{-t\phi(|\xi|^2)},\ \xi \in \mathbb{R}^d,\ t \geq 0.$$

Let $\phi \in \mathcal{CBF}$ without drift, $\phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(t)dt,\ \lambda > 0.$ The Lévy measure $\nu$ of $X$ has a radial density $j$, $\nu(dx) = j(|x|)dx$, given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(t)\, dt.$$

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Let there exist constants $0 < \delta_1 \leq \delta_2 < 1 \wedge \frac{d}{2}$ and $a_1, a_2 > 0$ such that

$$a_1 \lambda^{\delta_1} \leq \frac{\phi(\lambda x)}{\phi(x)} \leq a_2 \lambda^{\delta_2}, \quad \lambda \geq 1, x > 0. \tag{1}$$

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Assume that there are constants $C > 0$ and $\beta > 1$ such that

$$F(x,y) \leq C \frac{\Phi(|x-y|)^\beta}{1 + \Phi(|x|)^\beta + \Phi(|y|)^\beta}, \qquad \text{for all } \ x, y \in \mathbb{R}^d, \qquad (2)$$

where $\Phi(\lambda) := \phi(\lambda^{-2})^{-1}$, $\lambda > 0$.

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Assume that there are constants $C > 0$ and $\beta > \alpha$ such that

$$F(x,y) \leq C \frac{|x-y|^\beta}{1 + |x|^\beta + |y|^\alpha}, \qquad \text{for all } \ x, y \in \mathbb{R}^d, \tag{2}$$

# Main theorem

## Theorem 1

*Suppose that $X$ is the SBM via the subordinator whose Laplace exponent $\phi \in \mathcal{CBF}$ satisfies (1). Let $\beta > 1$ and assume that $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is bounded, symmetric and satisfies the condition (2). Let*

$$A_t^F = \sum_{0<s\leq t} F(X_{s-}, X_s).$$

*Then*

$$\mathbb{P}_x(A_\infty^F < \infty) = 1 \text{ for all } x \in \mathbb{R}^d \ \Rightarrow \ \sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_\infty^F] < \infty.$$

# Key elements of the proof of Theorem 1

1. Harnack inequality for $F$-harmonic functions of a class of SBMs satisfying the scaling condition (1);
2. Properties of the scaled process $X^R \overset{\mathrm{D}}{=} (R^{-1}X_{t/\phi(R^{-2})})_{t\geq 0}$;
3. Properties of the function $u(x) := \mathbb{E}_x[e^{-A^F_\infty}]$, $x \in \mathbb{R}^d$ and the proof of Theorem 1.

# Harnack inequality for $F$-harmonic functions

A non-negative function $u : \mathbb{R}^d \to [0, \infty)$ is *$F$-harmonic* in a bounded open set $D$ with respect to $X$ if for every open set $V \subset \overline{V} \subset D$ the following mean-value property holds:

$$u(x) = \mathbb{E}_x \left[ e^{-A^F_{\tau_V}} u(X_{\tau_V}) \right], \quad \text{for all } x \in V.$$

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## Theorem 2

*Let $D\subset\mathbb{R}^d$ be a bounded open set and $K\subset D$ a compact subset of $D$. Fix $\beta>1$ and $C>0$. There exists a constant $C_1=C_1(d,a_1,a_2,\delta_1,\delta_2,\beta,C,D,K)>1$ such that for every non-negative, symmetric function $F$ on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying*

$$F(x,y)\le C(\Phi(s)^\beta\wedge 1),$$

*and every $u:\mathbb{R}^d\to[0,\infty)$ which is $F$-harmonic with respect to $X$ in $D$, it holds that*

$$C_1^{-1}u(x)\le u(y)\le C_1u(x),\quad x,y\in K.$$

# Proof of the Harnack inequality

Let $B_r = B(0,r)$, $r > 0$, $\delta_{B_r}(x)$ the distance of $x$ to the boundary $\partial B_r$, and let $G_{B_r}$ denote the Green function of the process $X$ killed upon exiting $B_r$.
We use the following two-sided Green function estimate (Kim, Mimica, 2013) for all $r \in (0,1]$ and $x, y \in B_r$

$$G_{B_r}(x,y) \asymp \frac{\Phi(|x-y|)}{|x-y|^d}\left(1 \wedge \frac{\Phi(\delta_{B_r}(x))^{\frac{1}{2}}\Phi(\delta_{B_r}(y))^{\frac{1}{2}}}{\Phi(|x-y|)}\right),$$

where the constants depend only on $d$, $a_1$, $a_2$, $\delta_1$, $\delta_2$ from (1).

# Key Lemma in the proof of HI

## Lemma 3

*(i) There exists a constant* $C_2 = C_2(a_1, a_2, \delta_1, \delta_2, d)$ *such that for every* $r \in (0, 1]$,

$$\frac{G_{B_r}(x,y)G_{B_r}(z,w)}{G_{B_r}(x,w)} \leq \begin{cases} \dfrac{C_2\,\Phi(|x-y|)\Phi(|z-w|)}{\Phi(|x-w|)}\,\dfrac{|x-w|^d}{|x-y|^d|z-w|^d}, & (x,w)\in E_r, \\ C_2\,\Phi(|x-y|)^{\frac{1}{2}}\,\Phi(|z-w|)^{\frac{1}{2}}\,\dfrac{|x-w|^d}{|x-y|^d|z-w|^d}, & (x,w)\notin E_r, \end{cases}$$

*where* $E_r = \{(x,w) \in B_r \times B_r : |x-w| \leq \frac{1}{2}\max\{\delta_{B_r}(x), \delta_{B_r}(w)\}\}$.

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*where* $E_r = \{(x,w) \in B_r \times B_r : |x-w| \le \frac{1}{2}\max\{\delta_{B_r}(x), \delta_{B_r}(w)\}\}$.
*(ii) Let* $\beta > 1$, $C > 0$. *For every* $\varepsilon > 0$ *there exists a constant* $r_0 = r_0(d, a_1, a_2, \delta_1, \delta_2, \beta, C, \varepsilon) \in (0, 1]$ *such that for every* $r \in (0, r_0]$ *and every* $x, w \in B_r$,

$$\mathbb{E}_x^w\left[A^F_{\tau_{B_r}}\right] = \int_{B_r}\int_{B_r} \frac{G_{B_r}(x,y)G_{B_r}(z,w)}{G_{B_r}(x,w)}|F(y,z)|j(|y-z|)\,dz\,dy < \varepsilon.$$

# Proof of the Harnack inequality

Lemma 3 implies that for every $r \leq r_0$ and for $x, w \in B_r$

$$0 \leq \mathbb{E}_x^w \left[ A^F_{\tau_{B_r}} \right] < \log(2) \;\Rightarrow\; \frac{1}{2} \leq \mathbb{E}_x^w \left[ e^{-A^F_{\tau_{B_r}}} \right] \leq 1.$$

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This inequality implies

$$\frac{1}{2}\mathbb{E}_y[u(X_{\tau_{B_r}})] \leq u(y) \leq \mathbb{E}_y[u(X_{\tau_{B_r}})], \ y \in B_r.$$

The proof of the HI now follows by using the estimates for the Poisson kernel $P_{B_r}$ (Kim, Song, Vondraček, 2015.)

# Scaled process $X^R$

For each $R > 0$ define

$$\phi^R(s) = \frac{\phi(R^{-2}s)}{\phi(R^{-2})}, \quad s > 0$$

and note that $\phi^R \in \mathcal{CBF}$ satisfies the scaling condition (1) with same constants. Denote by $X^R$ the subordinate Brownian motion with the characteristic exponent $\psi^R(\xi) = \phi^R(|\xi|^2)$, $\xi \in \mathbb{R}^d$, and note that

$$(X_t^R)_{t\geq 0} \overset{\mathrm{D}}{=} (R^{-1}X_{t/\phi(R^{-2})})_{t\geq 0}. \tag{3}$$

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For $u : \mathbb{R}^d \to [0,\infty)$, $F : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$, $D \subset \mathbb{R}^d$, and any $R > 0$, set

$$u_R(x) := u(Rx), \quad F_R(x,y) := F(Rx,Ry), \quad D_R := \{Rx : x \in D\}.$$

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## Lemma 4

*Let $D$ be a bounded open set in $\mathbb{R}^d$, $R > 0$. If $u$ is regular $F$-harmonic in $D_R$ for $X$, then $u_R$ is regular $F_R$-harmonic in $D$ for $X^R$.*

# Properties of $F_R$ and $\hat{F}_R$

Assume that $F : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ is symmetric, bounded and satisfies (2).

(a) Then $F_R$ is symmetric, bounded and satisfies $F_R(x, y) \leq C\Phi^R(|x - y|)^\beta$ for $|x| \geq 1$ or $|y| \geq 1$.

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(b) For a bounded open set $D \subset B(0, 1)^c$ and $R \ge 1$ let

$$\widehat{F}_R(x, y) = \begin{cases} F_R(x, y) & \text{if } \ (x, y) \in (D \times \mathbb{R}^d) \cup (\mathbb{R}^d \times D) \\ 0 & \text{otherwise.} \end{cases}$$

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Then $\widehat{F}_R$ is symmetric, bounded and satisfies for $|x| \ge 1$ or $|y| \ge 1$

$$\begin{aligned} F_R(x, y) = F(Rx, Ry) &\le C\frac{\Phi(|Rx - Ry|)^\beta}{1 + \Phi(|Rx|)^\beta + \Phi(|Ry|)^\beta} \\ &= C\frac{\Phi^R(|x - y|)^\beta}{\Phi(R)^{-\beta} + \Phi^R(|x|)^\beta + \Phi^R(|y|)^\beta} \le C\Phi^R(|x - y|)^\beta, \end{aligned}$$

i.e. $\widehat{F}_R(x, y) \le C\Phi^R(|x - y|)^\beta$ for all $x, y \in \mathbb{R}^d$.

# Proof of Theorem 1:

Define $u(x) := \mathbb{E}_x[e^{-A^F_\infty}]$, $x \in \mathbb{R}^d$ and note that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A^F_\infty] < \infty \Leftrightarrow \inf_{x\in\mathbb{R}^d} u(x) = c, \text{ for some } c > 0 \overset{u \text{ is cont.}}{\Leftrightarrow} \liminf_{|x|\to\infty} u(x) > 0.$$

Let $M > 0$ and $D = V(0, 1, 2M + 1) = \{x \in \mathbb{R}^d : 1 < |x| < 2M + 1\}$ and $R \geq 1$. Since $u$ is regular $F$-harmonic in $D_R$ for $X$, it follows that $u_R$ is regular $\hat{F}_R$-harmonic in $D$ for $X^R$, where

$$\hat{F}_R = F_R \cdot 1_{(D\times\mathbb{R}^d)\cup(\mathbb{R}^d\times D)}.$$

By the HI it follows that there exists $c = c(d, a_1, a_2, \delta_1, \delta_2, \beta, C, M) > 1$ such that for all $R \geq 1$

$$c^{-1}u_R(y) \leq u_R(x) \leq cu_R(y) \quad \text{for all} \quad x, y \in \overline{V}(0, 2, 2M)\,.$$

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By the HI it follows that there exists $c = c(d, a_1, a_2, \delta_1, \delta_2, \beta, C, M) > 1$ such that for all $R \geq 1$

$$c^{-1}u(y) \leq u(x) \leq cu(y) \quad \text{for all} \quad x, y \in \overline{V}(0, 2R, 2RM).$$

# Proof of Theorem 1:

Suppose that there exists a sequence $(x_n)_{n\geq 1}$ in $\mathbb{R}^d$ such that $|x_n| \to \infty$ and $\lim_{n\to\infty} u(x_n) = 0$. Then there exists an increasing sequence $(k_n)_{n\geq 1}$ such that $x_n \in V_n := \overline{V}(0, 2^{k_n}, 2^{k_n}M)$ for every $n \geq 1$. One can choose $M \geq 2$ such that $X$ hits infinitely many sets $V_n$ $\mathbb{P}_x$-a.s. Hence, for $\mathbb{P}_x$-a.e. $\omega$ there exists a subsequence $(n_l = n_l(\omega))$ and a sequence of times $(t_l = t_l(\omega))$ such that $X_{t_l}(\omega) \in V_{n_l}$. Therefore it follows from the HI,

$$c^{-1}u(X_{t_l}(\omega)) \leq u(x_{n_l}) \leq cu(X_{t_l}(\omega)),$$

which implies that $\lim_{l\to\infty} u(X_{t_l}(\omega)) = 0$. But this is a contradiction with $\lim\limits_{t\to\infty} u(X_t) = 1$ $\mathbb{P}_x$-a.s. Therefore, $\sup_{x\in\mathbb{R}^d} \mathbb{E}_x[A_\infty^F] < \infty$ holds. □

Thank you for your attention.