Random difference equation and regularly varying tails

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Goal: Investigate $\mathbb{P}[X > t]$ as $t \to \infty$.

Kesten-Goldie: If for some $\alpha > 0$, $\mathbb{E}[A^\alpha] = 1$, $\mathbb{E}[B^\alpha] < \infty$ and $(\dots)$ then

$$\mathbb{P}[X > t] \sim ct^{-\alpha} \qquad \text{as } t \to \infty$$

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Grincevičius-Grey: If for some $\alpha > 0$, $\mathbb{E}[A^\alpha] < 1$, $\mathbb{E}[A^{\alpha+\varepsilon}] < \infty$ and $\mathbb{P}[B > t] \sim t^{-\alpha}L(t)$, ($L$ – slowly varying), then

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Question: What if $\mathbb{E}A^\alpha = 1$ and $\mathbb{P}[B > t] \sim t^\alpha L(t)$?

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From now on assume that $\mathbb{P}[B > t] = O(\mathbb{P}[A > t])$ and

$$\mathbb{P}[A > t] \sim c_A \log(t)^{-\beta} t^{-\alpha}, \quad \beta > 1,\ \alpha > 0.$$

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we have

$$\mathbb{P}[A_1 A_2 > t] = 2\mathbb{P}\left[A_1 A_2 > t,\ A_1 \le \sqrt{t}\right] + o\left(\log(t)^{-\beta} t^{-\alpha}\right)$$

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Similarly, if Y is independent of A, $\mathbb{P}[Y > t] \sim c_Y \mathbb{P}[A > t]$, then

$$\mathbb{P}[AY > t] \sim \left(\mathbb{E}[A^\alpha]c_Y + \mathbb{E}[Y^\alpha]\right)\mathbb{P}[A > t].$$

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For example, by induction

$$\mathbb{P}[A_1 \ldots A_n > t] \sim n\mathbb{E}[A^\alpha]^{n-1}\mathbb{P}[A > t].$$

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Proof.

$$\begin{aligned}\mathbb{P}[AY + B > t] = {} & \mathbb{P}\left[AY + B > t,\ Y \le \sqrt{t}\right] + \\ & \mathbb{P}\left[AY + B > t,\ A \vee B \le \frac{1}{2}\sqrt{t}\right] + o\left(\log(t)^{-\beta} t^{-\alpha}\right)\end{aligned}$$

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STEP 2: Write

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Assume $A, B \geq 0$, $\mathbb{E}[A^\alpha] < 1$, $\mathbb{P}[B > t] = O(\mathbb{P}[A > t])$ and $\mathbb{P}[A > t] \sim c_A \log(t)^{-\beta} t^{-\alpha}$ for $\beta > 1$, $\alpha > 0$.

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Then, for $\mu_\pm = \mathbb{E}A_\pm^\alpha$,

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$$\mathbb{P}[B > t] \sim c_B \mathbb{P}[A > t], \quad \mathbb{P}[C > t] \sim c_C \mathbb{P}[A > t].$$

Then $f_-(x) = 0$ and

$$f_+(x) = x^\alpha + c_C x^{\alpha/2} \log^+(x)^\alpha + c_B.$$

Whence, for

$$X \stackrel{d}{=} AX + C\sqrt{X}\log^+(X) + B, \quad X \text{ independent of } (A, B, C)$$

Example

If

$$\Psi(x) = Ax + C\sqrt{x}\log^+(x) + B$$

with $A, B, C \geq 0$ independent with

$$\mathbb{P}[B > t] \sim c_B \mathbb{P}[A > t], \quad \mathbb{P}[C > t] \sim c_C \mathbb{P}[A > t].$$

Then $f_-(x) = 0$ and

$$f_+(x) = x^\alpha + c_C x^{\alpha/2} \log^+(x)^\alpha + c_B.$$

Whence, for

$$X \stackrel{d}{=} AX + C\sqrt{X}\log^+(X) + B, \quad X \text{ independent of } (A, B, C)$$

we get

$$\mathbb{P}[X > t] \sim \frac{\mathbb{E}[X^\alpha + c_C X^{\alpha/2}\log^+(X)^\alpha] + c_B}{1 - \mathbb{E}[A^\alpha]} \mathbb{P}[A > t].$$