A hierarchical renormalization model on trees

Yueyun Hu (Paris 13)

The 3rd conference Probability and Analysis
15–19 May 2017, Będlewo, Poland

Overview

Model (Derrida and Retaux) :

Let X_0 be random variable taking values in $\{0, 1, 2, ...\}$. Define

$$X_{n+1} \stackrel{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of X_n.

Question :

What can we say about the asymptotic behaviors of X_n as $n \to \infty$?

Overview

Model (Derrida and Retaux) :

Let X_0 be random variable taking values in $\{0, 1, 2, ...\}$. Define

$$X_{n+1} \stackrel{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

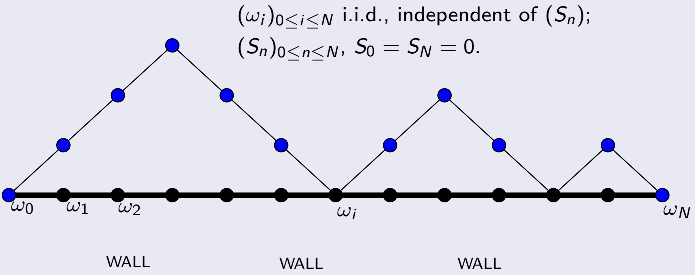
with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of X_n.

Question :

What can we say about the asymptotic behaviors of X_n as $n \to \infty$?

Outline

The pining model on $\mathbb{Z}$



The pining model on $\mathbb{Z}$

- The measure of polymer of length N :

$$P_{N,\omega}(dS) := \frac{1}{Z_N} \exp\left(\sum_{i=1}^{N} \omega_i 1_{\{S_i=0\}} \right),$$

- Z_N is called the partition function [$\omega_i > 0$ (i attractive) ; $\omega_i < 0$ (i repulsive)].
- See Giacomin's book (Random Polymer Models, 2007).

The pining model on $\mathbb{Z}$

- The measure of polymer of length N :

$$P_{N,\omega}(dS) := \frac{1}{Z_N} \exp\Big(\sum_{i=1}^{N} \omega_i 1_{\{S_i=0\}}\Big),$$

- Z_N is called the partition function [$\omega_i > 0$ (i attractive) ; $\omega_i < 0$ (i repulsive)].
- See Giacomin's book (Random Polymer Models, 2007).

The pining model on $\mathbb{Z}$

- The measure of polymer of length N :

$$P_{N,\omega}(dS) := \frac{1}{Z_N} \exp\Big(\sum_{i=1}^{N} \omega_i 1_{\{S_i=0\}}\Big),$$

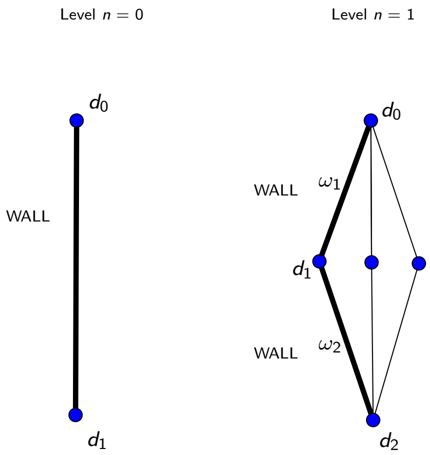
- Z_N is called the partition function [$\omega_i > 0$ (i attractive) ; $\omega_i < 0$ (i repulsive)].
- See Giacomin's book (Random Polymer Models, 2007).

Pinning model on a hierarchical lattice

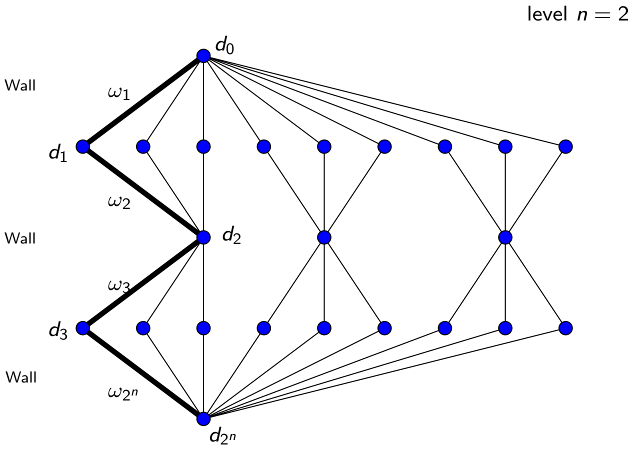
Derrida, Hakim and Vannimenus (1992)

1. At level 0, there is a unique bond.
2. Fix an integer $B \geq 2$ (for e.g. $B = 3$)
3. Rule : Each bond gives B branches consisting of 2 bonds each.

Case $B = 3$



Case $B = 3$


level $n = 2$
d_0
Wall
ω_1
d_1
ω_2
Wall
d_2
ω_3
d_3
Wall
ω_{2^n}
d_{2^n}

Pinning model on a hierarchical lattice

- At level n, each (direct) trajectory $(S_i)_{1\le i\le 2^n}$ (from d_0 to d_{2^n}) contains 2^n bonds.
- Choose the uniform measure $P_{B,n}$ on all possible trajectories [simple random walk $(S_i)_{1\le i\le 2^n}$ on the hierarchical lattice].
- Let $(\omega_i)_{1\le i\le 2^n}$ be i.i.d. and independent of (S_n).
- The partition function

$$Z_n := E_{B,n} \exp\Big(\sum_{i=1}^{2^n} \omega_i 1_{\{S_{i-1}=d_{i-1}, S_i=d_i\}}\Big),$$

where the expectation is only taken with respect to (S_n).

Pinning model on a hierarchical lattice

- Let $N_n :=$ number of trajectories γ from d_0 to d_{2^n} and

$$R_n := \sum_{\gamma:\gamma_0=d_0,\gamma_{2^n}=d_{2^n}} \exp\Big(\sum_{i=1}^{2^n} \omega_i 1_{\{\gamma_{i-1}=d_{i-1},\gamma_i=d_i\}}\Big).$$

- Then $Z_n = \frac{R_n}{N_n}$.
- Easy to see that

$$\begin{aligned} N_{n+1} &= B\, N_n^2, \\ R_{n+1} &= R_n^{(1)} R_n^{(2)} + (B-1) N_n^2, \end{aligned}$$

with two independent copies $R_n^{(1)}$, $R_n^{(2)}$ of R_n.

Pinning model on a hierarchical lattice

- Then
$$Z_{n+1} = \frac{R_{n+1}}{N_{n+1}} = \frac{Z_n^{(1)} Z_n^{(2)} + B - 1}{B},$$
with two independent copies $Z_n^{(1)}$, $Z_n^{(2)}$ of Z_n.
- Stability : $(B, Z) \mapsto (B', Z')$ with $B' := \frac{B}{B-1}, Z' := \frac{Z}{B-1}$.
- See Monthus and Garet (2008), Derrida, Giacomin, Lacoin and Toninelli (2009), Lacoin and Toninelli (2009), Giacomin, Lacoin and Toninelli (2010, 2011), Berger and Toninelli (2013) for the studies of this model [disorder relevance, critical line...]

Pinning model on a hierarchical lattice

- Let $X_n := \log Z_n$. Then

$$\begin{aligned} X_{n+1} &= \log Z_{n+1} \\ &= \log \frac{e^{(X_n^{(1)}+X_n^{(2)})} + B - 1}{B} \\ &= X_n^{(1)} + X_n^{(2)} + \log \frac{1 + (B-1)e^{-(X_n^{(1)}+X_n^{(2)})}}{B}. \end{aligned}$$

- Let $1 < B < 2$ and define $a := -\log(B-1) > 0$.
- If $X_n \geq -a$, a.s., then $X_{n+1} \geq -a$ a.s.

Derrida and Retaux (2014)'s model

- Fix $a > 0$. For any $n \geq 0$,

$$X_{n+1} \stackrel{\text{law}}{=} \max(X_n^{(1)} + X_n^{(2)}, -a),$$

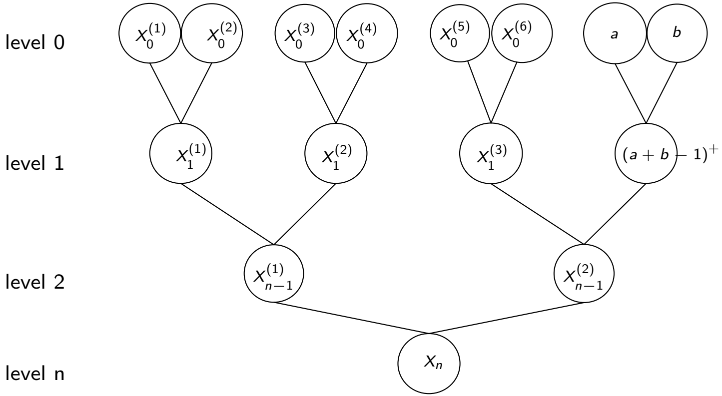
with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of X_n.

- Replacing X_n by $X_n + a$ and taking $a = 1$, the recursive equation becomes

$$X_{n+1} \stackrel{\text{law}}{=} (X_n^{(1)} + X_n^{(2)} - 1)^+, \qquad \forall n \geq 0,$$

with two independent copies $X_n^{(1)}$, $X_n^{(2)}$ of X_n.

Derrida and Retaux (2014)'s model



Derrida and Retaux (2014)'s model

1. Free energy : $F_\infty := \lim_{n\to\infty} \frac{\mathbb{E}(X_n)}{2^n} \in [0,\infty)$ exists.

   Proof : As $X_n \stackrel{\text{law}}{=} (X^{(1)}_{n-1} + X^{(2)}_{n-1} - 1)^+$, $2\mathbb{E}(X_{n-1}) \geq \mathbb{E}(X_n) \geq 2\mathbb{E}(X_{n-1}) - 1$, *implying that*

$$F_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{2^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - 1}{2^n}.$$

2. "Percolation on tree" : Let

$$X_0 \stackrel{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{Y\}},$$

   with $0 \leq p \leq 1$ and $Y > 0$ a positive random variable. Define

$$p_c := \sup\{0 \leq p \leq 1 : F_\infty(p) = 0\}.$$

Derrida and Retaux (2014)'s model

1. Free energy : $F_\infty := \lim_{n\to\infty} \frac{\mathbb{E}(X_n)}{2^n} \in [0,\infty)$ exists.
   Proof : As $X_n \stackrel{\text{law}}{=} (X_{n-1}^{(1)} + X_{n-1}^{(2)} - 1)^+$,
   $2\mathbb{E}(X_{n-1}) \geq \mathbb{E}(X_n) \geq 2\mathbb{E}(X_{n-1}) - 1$, *implying that*
   $$F_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{2^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - 1}{2^n}.$$

2. "Percolation on tree" : Let
   $$X_0 \stackrel{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{Y\}},$$
   with $0 \leq p \leq 1$ and $Y > 0$ a positive random variable. Define
   $$p_c := \sup\{0 \leq p \leq 1 : F_\infty(p) = 0\}.$$

Derrida and Retaux (2014)'s model

1. Free energy : $F_\infty := \lim_{n\to\infty} \frac{\mathbb{E}(X_n)}{2^n} \in [0,\infty)$ exists.
   Proof : As $X_n \overset{\text{law}}{=} (X_{n-1}^{(1)} + X_{n-1}^{(2)} - 1)^+$,
   $2\mathbb{E}(X_{n-1}) \geq \mathbb{E}(X_n) \geq 2\mathbb{E}(X_{n-1}) - 1$, *implying that*

$$F_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{2^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - 1}{2^n}.$$

2. "Percolation on tree" : Let

$$X_0 \overset{\text{law}}{=} (1-p)\delta_{\{0\}} + p\delta_{\{Y\}},$$

with $0 \leq p \leq 1$ and $Y > 0$ a positive random variable. Define

$$p_c := \sup\{0 \leq p \leq 1 : F_\infty(p) = 0\}.$$

Derrida and Retaux (2014)'s model

Question 1 : Value of p_c.

- Example : $Y \equiv 2$; $p_c = 1/2$?

Question 2 : Behavior of F_∞.

- If $p_c < 1$, when $p \downarrow p_c$, what is the behavior of $F_\infty(p)$?

Derrida and Retaux (2014)'s model

Question 1 : Value of p_c.

- Example : $Y \equiv 2$; $p_c = 1/2$?

Question 2 : Behavior of F_∞.

- If $p_c < 1$, when $p \downarrow p_c$, what is the behavior of $F_\infty(p)$?

Value of p_c

Theorem (Collet, Eckman, Glaser and Martin 1984)

Suppose that $Y \in \{1, 2, ...\}$. Then

$$p_c = \frac{1}{1 + \mathbb{E}((Y-1)2^Y)}.$$

As example, if $Y \equiv 2$, then $p_c = \frac{1}{5}$.

Open question

Find p_c for a general r.v. $Y \in \mathbb{R}_+$; or even for $Y \in \frac{1}{2}\mathbb{N}$...

Value of p_c

Theorem (Collet, Eckman, Glaser and Martin 1984)

Suppose that $Y \in \{1, 2, ...\}$. Then

$$p_c = \frac{1}{1 + \mathbb{E}((Y-1)2^Y)}.$$

As example, if $Y \equiv 2$, then $p_c = \frac{1}{5}$.

Open question

Find p_c for a general r.v. $Y \in \mathbb{R}_+$; or even for $Y \in \frac{1}{2}\mathbb{N}$...

Derrida and Retaux (2014)'s model

Corollary

For any general r.v. $Y \in \mathbb{R}_+$,

$$p_c > 0 \iff \mathbb{E}(Y2^Y) < \infty.$$

Open question

Find a probabilistic proof on the above $L \log L$-condition.

Derrida and Retaux (2014)'s model

Corollary

For any general r.v. $Y \in \mathbb{R}_+$,

$$p_c > 0 \iff \mathbb{E}(Y2^Y) < \infty.$$

Open question

Find a probabilistic proof on the above $L \log L$-condition.

Derrida's conjecture on the free energy

Derrida's conjecture :

If $p_c > 0$, then

$$F_\infty(p) = \exp\Big(- \frac{K + o(1)}{(p - p_c)^{1/2}} \Big), \qquad p \downarrow p_c,$$

for some explicit constant $K > 0$.

Why do we need $p_c > 0$?

Otherwise, the above conjecture is wrong.

Derrida's conjecture on the free energy

Derrida's conjecture :

If $p_c > 0$, then

$$F_\infty(p) = \exp\Big(- \frac{K + o(1)}{(p - p_c)^{1/2}} \Big), \qquad p \downarrow p_c,$$

for some explicit constant $K > 0$.

Why do we need $p_c > 0$?

Otherwise, the above conjecture is wrong.

Model

Model (Galton-Watson case)

Let ν be an integer-valued random variable such that $m := \mathbb{E}(\nu) \in (1, \infty)$. Consider the recursive equation

$$X_{n+1} \overset{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+,$$

where $X_n^{(1)}, X_n^{(2)}, \ldots,$ are i.i.d. copies of X_n, and independent of ν.

Suppose $X_0 \overset{\text{law}}{=} (1-p)\delta_0 + p\delta_Y$, with $Y > 0$ a.s. (not necessarily integer-valued). Let

$$F_\infty(p) := \lim_{n\to\infty} \frac{1}{m^n}\mathbb{E}(X_n) \in [0, \infty)$$

and define

$$p_c := \sup\{0 \le p \le 1 : F_\infty(p) = 0\}.$$

Open question

What is the value of p_c in the Galton-Watson case ?

Only known when ν equals some integer a.s. [Collet et al. (1984)].

Suppose $X_0 \stackrel{\text{law}}{=} (1-p)\delta_0 + p\delta_Y$, with $Y > 0$ a.s. (not necessarily integer-valued). Let

$$F_\infty(p) := \lim_{n\to\infty} \frac{1}{m^n}\mathbb{E}(X_n) \in [0,\infty)$$

and define

$$p_c := \sup\{0 \le p \le 1 : F_\infty(p) = 0\}.$$

Open question

What is the value of p_c in the Galton-Watson case ?
Only known when ν equals some integer a.s. [Collet et al. (1984)].

Proposition

In the Galton-Watson case, $p_c = 0$ if and only if $\mathbb{E}(Ym^Y) = \infty$.

Proof :

"If part" is a consequence of Collet et al.'s theorem ;
"Only if" part : an argument by induction to control the generating function of X_n.

Proposition

In the Galton-Watson case, $p_c = 0$ if and only if $\mathbb{E}(Ym^Y) = \infty$.

Proof :

"If part" is a consequence of Collet et al.'s theorem ;
"Only if" part : an argument by induction to control the generating function of X_n.

Assumptions

We work under the assumption $\mathbb{E}(Ym^Y) = \infty$ (i.e. $p_c = 0$).
Assume that either

$$\mathbb{P}(Y > x) \approx x^\alpha m^{-x}, \qquad x \to \infty,$$

for some $\alpha \geq -2$,
or

$$\mathbb{P}(Y > x) \approx \theta^{-x}, \qquad x \to \infty,$$

for some $1 < \theta < m$. We study the rate of convergence of $F_\infty(p)$ as $p \to 0$.

Exponential decay

Theorem 1 (H., Shi (2017+))

Assume that

$$\mathbb{P}(Y > x) \approx x^{\alpha} m^{-x}, \qquad x \to \infty,$$

for some $\alpha > -2$. Then

$$F_\infty(p) = \exp\Big(- p^{-(1+o(1))/(2+\alpha)} \Big), \qquad p \to 0.$$

Polynomial decay

Theorem 2 (H., Shi (2017+))

Assume that
$$\mathbb{P}(Y > x) \approx \theta^{-x}, \qquad x \to \infty,$$
for some $1 < \theta < m$. Then
$$F_\infty(p) \approx p^{\frac{\log m}{\log(m/\theta)}}, \qquad p \to 0.$$

- From $X_{n+1} \stackrel{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+$, we get that

$$m\mathbb{E}(X_n) - 1 \le \mathbb{E}(X_{n+1}) \le m\mathbb{E}(X_n).$$

- Then

$$F_\infty(p) = \lim \uparrow m^{-n}(\mathbb{E}(X_n) - \frac{1}{m-1}) = \lim \downarrow m^{-n}\mathbb{E}(X_n).$$

- Hence $F_\infty(p) > 0$ if and only if $\exists n \ge 1$ such that $\mathbb{E}(X_n) > \frac{1}{m-1}$.
- Let $n_0 := \inf\{n \ge 1 : \mathbb{E}(X_n) > \frac{1}{m-1}\}$, then

$$F_\infty(p) \asymp m^{-n_0}.$$

- From $X_{n+1} \overset{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+$, we get that

$$m\mathbb{E}(X_n) - 1 \le \mathbb{E}(X_{n+1}) \le m\mathbb{E}(X_n).$$

- Then

$$F_\infty(p) = \lim \uparrow m^{-n}(\mathbb{E}(X_n) - \frac{1}{m-1}) = \lim \downarrow m^{-n}\mathbb{E}(X_n).$$

- Hence $F_\infty(p) > 0$ if and only if $\exists n \ge 1$ such that $\mathbb{E}(X_n) > \frac{1}{m-1}$.
- Let $n_0 := \inf\{n \ge 1 : \mathbb{E}(X_n) > \frac{1}{m-1}\}$, then

$$F_\infty(p) \asymp m^{-n_0}.$$

- From $X_{n+1} \stackrel{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+$, we get that

$$m\mathbb{E}(X_n) - 1 \le \mathbb{E}(X_{n+1}) \le m\mathbb{E}(X_n).$$

- Then

$$F_\infty(p) = \lim \uparrow m^{-n}(\mathbb{E}(X_n) - \frac{1}{m-1}) = \lim \downarrow m^{-n}\mathbb{E}(X_n).$$

- Hence $F_\infty(p) > 0$ if and only if $\exists n \ge 1$ such that $\mathbb{E}(X_n) > \frac{1}{m-1}$.
- Let $n_0 := \inf\{n \ge 1 : \mathbb{E}(X_n) > \frac{1}{m-1}\}$, then

$$F_\infty(p) \asymp m^{-n_0}.$$

- From $X_{n+1} \overset{\text{law}}{=} (\sum_{i=1}^{\nu} X_n^{(i)} - 1)^+$, we get that

$$m\mathbb{E}(X_n) - 1 \le \mathbb{E}(X_{n+1}) \le m\mathbb{E}(X_n).$$

- Then

$$F_\infty(p) = \lim \uparrow m^{-n}(\mathbb{E}(X_n) - \frac{1}{m-1}) = \lim \downarrow m^{-n}\mathbb{E}(X_n).$$

- Hence $F_\infty(p) > 0$ if and only if $\exists n \ge 1$ such that $\mathbb{E}(X_n) > \frac{1}{m-1}$.
- Let $n_0 := \inf\{n \ge 1 : \mathbb{E}(X_n) > \frac{1}{m-1}\}$, then

$$F_\infty(p) \asymp m^{-n_0}.$$

- Proof of the polynomial decay case : When

$$\mathbb{P}(X_0 > x) \sim p\,\theta^{-x}, \qquad x \to \infty,$$

with $1 < \theta < m$. Prove that

$$n_0 \approx \frac{\log 1/p}{\log m/\theta}, \qquad p \to 0,$$

where $n_0 := \inf\{n \geq 1 : \mathbb{E}(X_n) > \frac{1}{m-1}\}$.

Proof of Theorem 2 : Lower bound of $F_\infty(p)$, i.e. upper bound of n_0

- Let $M_n := \max_{1\le i\le m^n} X_0^{(i)}$. Then $X_n \ge (M_n - n)^+$.
- Extreme value theory implies that

$$M_n \approx \frac{1}{\log\theta}\,(n\log m - \log\frac{1}{p}).$$

- Then

$$\mathbb{E}(X_n) \ge \mathbb{E}(M_n) - n \approx (\frac{\log m}{\log\theta} - 1)n - \frac{\log\frac{1}{p}}{\log\theta} > \frac{1}{m-1},$$

for all $n > (\frac{1}{m-1} + \frac{\log\frac{1}{p}}{\log\theta})/(\frac{\log m}{\log\theta} - 1) \approx \frac{\log(1/p)}{\log m/\theta}$.

Consider the generating functions

$$G_n(s) := \mathbb{E}(s^{X_n}), \qquad h(s) := \mathbb{E}(s^{\nu}).$$

Then

$$G_{n+1}(s) = \frac{h(G_n(s))}{s} + \frac{s-1}{s}\, h(G_n(0)) \le \frac{h(G_n(s))}{s} + \frac{s-1}{s}\,.$$

This gives, by iteration, an upper bound on $G_n(s) - 1$. Choose $s = s(p) \sim m$ gives a lower bound for n_0, i.e. an upper bound for $F_\infty(p)$.

The proof of the upper bound in Theorem 1 is similar, except for the case $\alpha \in (-2, -1)$ which is a slightly more delicate.

Consider the generating functions

$$G_n(s) := \mathbb{E}(s^{X_n}), \qquad h(s) := \mathbb{E}(s^{\nu}).$$

Then

$$G_{n+1}(s) = \frac{h(G_n(s))}{s} + \frac{s-1}{s} h(G_n(0)) \leq \frac{h(G_n(s))}{s} + \frac{s-1}{s} .$$

This gives, by iteration, an upper bound on $G_n(s) - 1$. Choose $s = s(p) \sim m$ gives a lower bound for n_0, i.e. an upper bound for $F_\infty(p)$.
The proof of the upper bound in Theorem 1 is similar, except for the case $\alpha \in (-2, -1)$ which is a slightly more delicate.

THANK YOU