On the rate of convergence of the Biggins martingale in supercritical branching random walks

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What is the branching random walk and the Biggins martingale?

$\mathcal{M}$– a point process on $\mathbb{R}$; $L := \mathcal{M}(\mathbb{R})$, possibly infinite.

By branching random walk (BRW) on $\mathbb{R}$ is meant the sequence of point processes $(\mathcal{M}_n)_{n\in\mathbb{N}_0}$, where for any Borel set $B \subset \mathbb{R}$,

$$\mathcal{M}_0(B) := \mathbb{1}_{\{0\in B\}},$$

$$\mathcal{M}_{n+1}(B) := \sum_r \mathcal{M}_{n,r}(B - A_{n,r}), \quad n \in \mathbb{N}_0.$$

Here $(A_{n,r})$ are the points of $\mathcal{M}_n$, and $(\mathcal{M}_{n,r})$ are independent copies of $\mathcal{M}$.

Supercriticality : if $\mathbb{P}\{L < \infty\} = 1$ it is additionally assumed that $\mathbb{E}L > 1$.

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Supercriticality : if $\mathbb{P}\{L < \infty\} = 1$ it is additionally assumed that $\mathbb{E}L > 1$.

Assume that for some $\gamma > 0$

$$m(\gamma) := \mathbb{E}\int_{\mathbb{R}} e^{\gamma x}\mathcal{M}(\mathrm{d}x) \in (0, \infty)$$

and set

$$W_n := W_n(\gamma) = m(\gamma)^{-n}\int_{\mathbb{R}} e^{\gamma x}\mathcal{M}_n(\mathrm{d}x), \quad n \in \mathbb{N}.$$

The sequence $(W_n, \sigma(\mathcal{M}_1, \dots, \mathcal{M}_n))_{n\in\mathbb{N}}$ is a nonnegative martingale (Kingman (1975) and Biggins (1977)) which is called the Biggins martingale.

Uniform integrability of the Biggins martingale:

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Let $(M_k, Q_k)_{k\in\mathbb{N}}$ be independent copies of an $\mathbb{R}^2$-valued random vector (M, Q) with arbitrary dependence between M and Q.

If the series

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$\mathcal{M}$– a point process with points (X_i); $L := \mathcal{M}(\mathbb{R})$

Standing assumption: there exists $\gamma > 0$ such that $m(\gamma) := \mathbb{E} \sum_{i=1}^{L} e^{\gamma X_i} < \infty$.

Theorem (Alsmeyer & I. (2009))
The Biggins martingale is uniformly integrable if, and only if,

$$Z^* := Q_1^* + M_1^* Q_2^* + M_1^* M_2^* Q_3^* + \dots < \infty \quad \text{a.s.},$$

where

$$\mathbb{P}\{(M^*, Q^*) \in A\} = \mathbb{E} \sum_{i=1}^{L} \frac{e^{\gamma X_i}}{m(\gamma)} \mathbb{1}_A \left(\frac{e^{\gamma X_i}}{m(\gamma)}, \sum_{j=1}^{L} \frac{e^{\gamma X_j}}{m(\gamma)} \right)$$

for any Borel set A in $\mathbb{R}^2$.

Uniform integrability of the Biggins martingale:

Theorem (Alsmeyer & I. (2009))
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According to Goldie & Maller (2000), $Z^* < \infty$ a.s. if, and only if,

$$\lim_{n\to\infty} M_1^* M_2^* \cdot \ldots \cdot M_n^* = 0 \quad \text{a.s.}$$

and

$$\mathbb{E}J(\log^+ Q^*) = \mathbb{E}W_1 J(\log^+ W_1) < \infty,$$

where

$$J(x) := \frac{x}{\int_0^x \mathbb{P}\{-\log M^* > y\} \mathrm{d}y}, \quad x > 0.$$

In particular, if $\mathbb{E}\log M^* \in (-\infty, 0)$, then $\mathbb{E}W_1 \log^+ W_1 < \infty$ is a necessary and sufficient condition for uniform integrability of the Biggins martingale.

The rate of convergence:
CLT for the tail of the Biggins martingale

$m(\gamma) = \mathbb{E} \int_{\mathbb{R}} e^{\gamma x} \mathcal{M}(\mathrm{d}x)$, $\gamma > 0$;
$W_n(\gamma) = (m(\gamma))^{-n} \int_{\mathbb{R}} e^{\gamma x} \mathcal{M}_n(\mathrm{d}x)$, $n \in \mathbb{N}$;
$\lim_{n\to\infty} W_n(\gamma) = W_\infty(\gamma)$ a.s.

Theorem (I. & Kabluchko (2016))
Suppose that $m(1) = 1$, $\sigma^2 := \mathrm{Var}\, W_1(1) \in (0, \infty)$ and $m(2) < 1$.
Then, as $n \to \infty$,

$$\left(\frac{W_\infty(1) - W_{n+r}(1)}{(m(2))^{(n+r)/2}}\right)_{r\in\mathbb{N}_0} \overset{\text{f.d.}}{\Rightarrow} \left(\sqrt{v^2 W_\infty(2)}\, U_r\right)_{r\in\mathbb{N}_0},$$

where $v^2 := \mathrm{Var}\, W_\infty(1) = \sigma^2(1 - m(2))^{-1}$, and $(U_r)_{r\in\mathbb{N}_0}$ is a stationary zero-mean Gaussian sequence which is independent of $W_\infty(2)$ and has the covariance

$$\mathrm{Cov}\,(U_r, U_s) = (m(2))^{|r-s|/2}, \quad r, s \in \mathbb{N}_0.$$

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Corollary (I. & Kabluchko (2016))
Suppose that $m(1) = 1$, $\mathrm{Var}\, W_1(1) \in (0, \infty)$ and $m(2) < 1$. Then, as $n \to \infty$,

$$\frac{W_\infty(1) - W_n(1)}{(m(2))^{n/2}} \overset{\text{d}}{\to} \mathrm{normal}\,(0, v^2 W_\infty(2)).$$

The rate of convergence:
relevant literature

Here are several related results, with pointers to the literature.

- a CLT for the tail martingale of a Galton-Watson process – Athreya (1968) and Heyde (1970)
- a functional CLT for the tail martingale of a Galton-Watson process – Heyde & Brown (1971)
- CLT's for multitype branching processes – Kesten & Stigum (1966), Athreya (1968) and Asmussen & Keiding (1978)
- a CLT for the tail martingale of a weighted branching processes – Rösler, Topchii & Vatutin (2002)
- a CLT for the tail martingale of a complex-valued branching Brownian motion – Hartung & Klimovsky (2017+)
- CLT's for tail martingales associated with random trees – Neininger (2015), Grübel & Kabluchko (2016) and Sulzbach (2017)
- CLT's for branching diffusions and superprocesses – Adamczak & Miłoś (2015) and Ren, Song & Zhang (2015)

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Theorem (I. & Kabluchko (2016))
Assume that $m(1) = 1$, $\sigma^2 = \mathrm{Var}\, W_1(1) \in (0, \infty)$, $\mathbb{E} W_1(2) \log^+ W_1(2) < \infty$, and that the function $r \to (m(r))^{1/r}$ is finite and decreasing on $[1, 2]$ with

$$\frac{-\log m(2)}{2} < -\frac{m'(2)}{m(2)},$$

where m' denotes the left derivative. Then $W_\infty(1)$ and $W_\infty(2)$ are positive almost surely on the survival set, and

$$\limsup_{n\to\infty} \frac{W_\infty(1) - W_n(1)}{\sqrt{(m(2))^n \log n}} = \sqrt{2v^2 W_\infty(2)},$$
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almost surely, where $v^2 = \mathrm{Var}\, W_\infty(1) = \sigma^2 (1 - m(2))^{-1} < \infty$.

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Why is it a law of the iterated logarithm? In view of

$$\mathrm{Var}\,[W_\infty(1) - W_n(1)] = v^2 (m(2))^n$$

one may replace $\log n$ by the asymptotically equivalent expression $\log\log(v^2 (m(2))^n)$, whence the term.

The rate of convergence:
other results

$(W_n)_{n\in\mathbb{N}_0}$ – uniformly integrable Biggins martingale

W – its a.s. limit

For $a>0$ and $p>1$,
the L_p-convergence of $\sum_{n\geq 0} e^{an}(W-W_n)$ – Alsmeyer, I., Polotskiy & Rösler (2009)

The a.s. convergence of $\sum_{n\geq 0} e^{an}(W-W_n)$ – I.& Meiners (2010)

For a function $b:\mathbb{R}^+\to\mathbb{R}^+$ regularly varying at ∞ of index α, $\alpha>-1$, the a.s. convergence of $\sum_{n\geq 0} b(n)(W-W_n)$ – I. (2006)

The rate of convergence:

exponentially fast a.s. convergence for the tail of the Biggins martingale

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$\lim_{n\to\infty} W_n(\gamma) = W_\infty(\gamma)$ a.s.

Theorem (I. & Meiners (2010))
Let $a > 0$ be given. Assume that $m(1) = 1$,

$$e^a\, m^{1/r}(r) \leq 1 \qquad \text{for some } r \in (1, 2)$$

and define θ to be the minimal $r > 1$ such that $e^{ar} m(r) = 1$. Assume further that

$$\mathbb{E} W_1(1)^\theta \ < \ \infty,$$

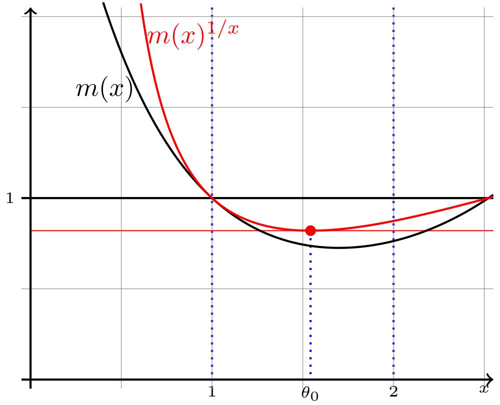
and in case when $a = -\log \inf_{r\geq 1}\ m^{1/r}(r)$ (which implies that $\theta = \theta_0$ satisfies $m^{1/\theta_0}(\theta_0) = \inf_{1\leq\theta\leq 2} m^{1/\theta}(\theta)$) assume that

$$-\ \frac{\log m(\theta_0)}{\theta_0} \ < \ -\ \frac{m'(\theta_0)}{m(\theta_0)}.$$

Then the Biggins martingale $(W_n(1))_{n\in\mathbb{N}_0}$ is uniformly integrable and $\sum_{n\geq 0} e^{an}(W_\infty(1) - W_n(1))$ converges almost surely.

The rate of convergence:
exponentially fast a.s. convergence for the tail of the Biggins martingale

A typical situation in which the previous theorem applies. The bottom point of the graph of $m^{1/x}(x)$ is marked by a filled red circle. The vertical dashed blue line connects this point to the x-axis indicating the point θ_0. The red horizontal line and the black horizontal line at 1 indicate the open interval of possible values of e^{-a} such that $a > 0$ and $e^a m^{1/\theta_0} m(\theta_0) < 1$. For those a's, $\sum_{n\geq 0} e^{an}(W_\infty(1) - W_n(1))$ converges a.s.



THANK YOU FOR YOUR ATTENTION!