Passage times of additive random walks and enumeration formulas of multitype forests.

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Multitype type branching processes

$\nu = (\nu_1, \ldots, \nu_d)$ with ν_i, $i \in [d]$ distributions on $\mathbb{Z}_+^d$.

$$\mathbf{Z} = (Z^{(1)}, \ldots, Z^{(d)})$$

d-type branching process with progeny distribution ν.

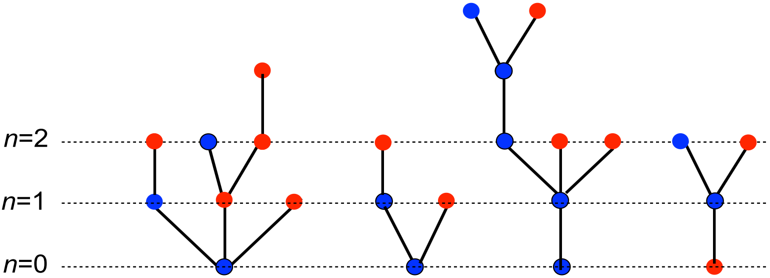
For $\mathbf{k} = (k_1, \ldots, k_d)$ and $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{Z}_+^d$,

$$\mathbb{P}(\mathbf{Z}(1) = \mathbf{k} \,|\, \mathbf{Z}(0) = \mathbf{r}) = \nu_1^{*r_1} * \cdots * \nu_d^{*r_d}(k_1, \ldots, k_d)\,.$$

We assume that ν is irreducible, critical or sub-critical, so that in particular,

$$\inf\{n : \mathbf{Z}(n) = 0\} < \infty\,, \quad \text{a.s.}$$

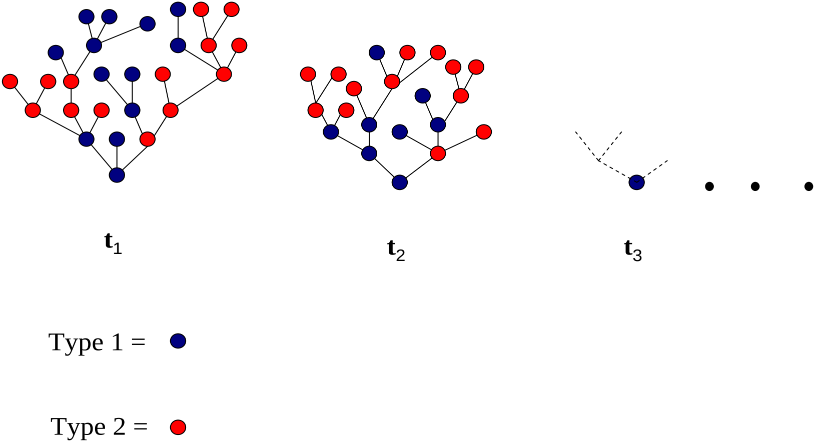
Multitype type branching processes



$Z^{(i)}(n)$, number of individuals of type i, at generation n in the forest :

$$\text{for } n = 2, \;\; \mathbf{Z}(2) = (Z^{(1)}(2), Z^{(2)}(2)) = (3, 5)\,.$$

Encoding 2-type forests



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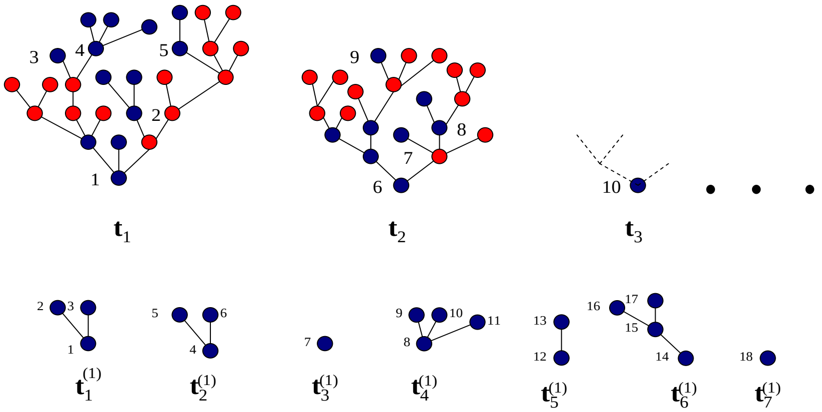

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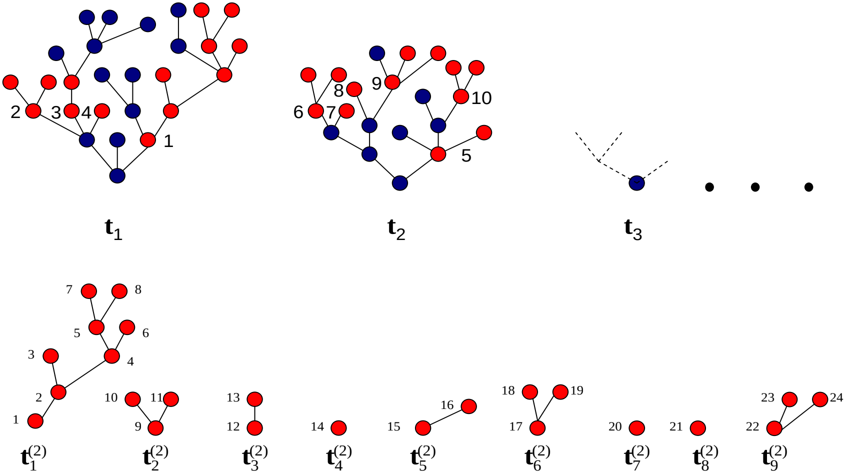

Encoding 2-type forests

Let $k_i(u)$ be the number of children of type i of the vertex u.

Let $u_n^{(i)}$ be the n-th vertex in the forest of type i.

Then define $X^{(1)} = (X^{1,1}, X^{1,2})$ and $X^{(2)} = (X^{2,1}, X^{2,2})$ by :

$$X_{n+1}^{1,1} - X_n^{1,1} = k_1(u_n^{(1)}) - 1 \qquad X_{n+1}^{2,1} - X_n^{2,1} = k_1(u_n^{(2)})$$

$$X_{n+1}^{1,2} - X_n^{1,2} = k_2(u_n^{(1)}) \qquad X_{n+1}^{2,2} - X_n^{2,2} = k_2(u_n^{(2)}) - 1 \, .$$

Proposition

The chains $X^{(1)}$ and $X^{(2)}$ are independent random walks in $\mathbb{Z} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \mathbb{Z}$, respectively, with step distributions :

$$P(X_1^{(1)} = (i,j)) = \nu_1(i+1, j) \, , \quad P(X_1^{(2)} = (i,j)) = \nu_2(i, j+1) \, .$$

$X^{(i)} = (X^{i,j}(n), 1 \le j \le d,\ n \ge 0)$, $i \in [d]$, independent random walks :

- $X^{i,j}$, for $i \neq j$ are $\mathbb{Z}_+$ valued and nondecreasing
- $X^{i,i}$ is $\mathbb{Z}$ valued and downward skip free : $X^{i,i}(n+1) - X^{i,i}(n) \ge -1$.

Proposition

The branching process $\mathbf{Z}$, *with* $\mathbf{Z}(0) = \mathbf{r}$, *with satisfies*

$$Z^{(j)}(n) = r_j + \sum_{i=1}^{d} X^{i,j}\left(\sum_{k=0}^{n-1} Z^{(i)}(k)\right), \quad n \ge 1, \quad j \in [d].$$

Continuous time : Chaumont ('15) – Caballero, Perez, Uribe ('17)

Since $Z^{(j)}(\infty) = 0$, we obtain :

$$0 = r_j + \sum_{i=1}^{d} X^{i,j}\left(\sum_{k=0}^{\infty} Z^{(i)}(k)\right)$$

so that the total population,

$$\mathbf{T}(\mathbf{r}) := \left(\sum_{k=0}^{\infty} Z^{(1)}(k), \ldots, \sum_{k=0}^{\infty} Z^{(d)}(k) \right)$$

is the first passage time at $\mathbf{r} = (r_1, \ldots, r_d)$ of the additive random walk :

$$\mathbf{n} = (n_1, \ldots, n_d) \;\mapsto\; \mathbf{X}(\mathbf{n}) = \sum_{i=1}^{d} X^{(i)}(n_i)\,,$$

that is,

$$\mathbf{T}(\mathbf{r}) = \inf\left\{\mathbf{n} : \mathbf{X}(\mathbf{n}) = -\mathbf{r}\right\}\,.$$

Recall that for $\mathbf{r} = (r_1, \ldots, r_d)$,

$$T(\mathbf{r}) = \inf \left\{ \mathbf{n} : \sum_{i=1}^{d} X^{(i)}(n_i) = -\mathbf{r} \right\} .$$

Theorem (Multivariate Ballot theorem)

Let $\mathbf{r} \in \mathbb{Z}_+^d$, $\mathrm{n} \in \mathbb{N}^d$, $k_{ii} \leq 0$ *and* $k_{ij} \geq 0$ *if* $i \neq j$ *be such that*

$$r_j + \sum_{i=1}^{d} k_{ij} = 0 , \quad j = 1, \ldots, d ,$$

then

$$P(X^{i,j}(n_i) = k_{ij}, T(\mathbf{r}) = \mathbf{n}) = \frac{\mathit{det}(-k_{ij})}{n_1 n_2 \ldots n_d} P(X^{i,j}(n_i) = k_{ij}) .$$

Total population of multitype type branching processes

For a multitype forest with

- r_i trees whose root is of type i,
- n_i vertices of type i,

$\mathbf{r} = (r_1, \dots, r_d)$ and $\mathbf{n} = (n_1, \dots, n_d)$ satisfy the system :

$$r_j + \sum_{i=1}^{d} X^{i,j}(n_i) = 0 \,, \quad j = 1, \dots, d \,.$$

- Moreover, $\mathbf{n} = \inf \{\mathbf{k} : \mathbf{X}(\mathbf{k}) = -\mathbf{r}\}$.

Total population of multitype type branching processes

Consider a forest with r_i, roots of type $i = 1, \dots, d$ and define :

- O_i total offspring of type i.
- $A_{ij} = X^{i,j}(n_i)$ the number of individuals of type j whose parent is of type i.

$$\{O_1 = n_1, \dots, O_d = n_d, A_{ij} = k_{ij}, 1 \le i \ne j \le d\}$$

$$= \{X^{i,j}(n_i) = k_{ij} \text{ and } \mathbf{n} = \inf\{\mathbf{k} : \mathbf{X}(\mathbf{k}) = -\mathbf{r}\}\}.$$

Total population of multitype type branching processes

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Theorem

For all integers $n_i \geq r_i \geq 1$, $i = 1, \dots, d$, $k_{ij} \geq 0$,

$$\begin{aligned} & \mathbb{P}_{(r_1,\dots,r_d)}(O_1 = n_1, \dots, O_d = n_d, A_{ij} = k_{ij}, 1 \leq i \neq j \leq d) \\ = \quad & \frac{\mathit{det}(-k_{ij})}{n_1 n_2 \dots n_d} \prod_{i=1}^{d} \nu_i^{*n_i}(k_{i1}, \dots, k_{i,i-1}, n_i - k_{ii}, k_{i,i+1}, \dots, k_{id}) \, . \end{aligned}$$

Application to enumeration of forests

Let $\mathcal{F}_d^{k_{ij},\mathrm{n}}$ be the set of plane forests with n_i vertices of type i, r_i roots of type i and such that for $i \neq j$, k_{ij} vertices of type j have a parent of type i.

Theorem

$$\left|\mathcal{F}_d^{k_{ij},\mathrm{n}}\right| = \frac{\det(-k_{ij})}{n_1 n_2 \dots n_d} \prod_{i,j=1}^{d} \binom{n_i + k'_{ij} - 1}{k'_{ij}} ,$$

where $k'_{ii} = n_i + k_{ii}$ *and for* $i \neq j$, $k'_{ij} = k_{ij}$.

Case $d = 1$ due to Pitman (1997).

Let F be a d-type branching forest with progeny law ν given by

$$\nu_i(k_1, \cdots, k_d) = \prod_{j=1}^{d} (1-p)^{k_j} p, \quad i = 1, 2, \ldots d,$$

Then

$$\mathbb{P}_{\mathrm{r}}\left(\mathrm{F} = \mathbf{f}\right) = \prod_{u \in \mathbf{f}} \nu_{\mathsf{type}(u)}(\mathsf{children}(u)) = \prod_{i,j=1}^{d} (1-p)^{k'_{ij}} \, p^{n_i}.$$

Since this probability is the same for all the forests $\mathbf{f} \in \mathcal{F}_d^{k_{ij},\mathrm{n}}$, the following distribution is the uniform distribution on $\mathcal{F}_d^{k_{ij},\mathrm{n}}$:

$$\mathbb{P}_{\mathrm{r}}\Big(\mathrm{F} = \mathbf{f} \;\Big|\, O(F) = \mathbf{n}, A_{ij}(F) = k_{ij}, i, j \in [d], i \neq j\Big) = \Big|\mathcal{F}_d^{k_{ij},\mathrm{n}}\Big|^{-1}.$$

But Theorem on the total population tells us that

$$\mathbb{P}_{\mathrm{r}}\Big(O(F) = \mathbf{n}, A_{ij} = k_{ij}, i, j \in [d], i \neq j\Big) =$$

$$\frac{\det(-k_{ij})}{n_1 n_2 \ldots n_d} \prod_{i,j=1}^{d} \binom{n_i + k'_{ij} - 1}{k'_{ij}} \left(\prod_{i,j=1}^{d} (1-p)^{k'_{ij}} p^{n_i}\right).$$

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Application to enumeration of forests

For $\mathbf{c} = (c_{i,j,k})_{i,j\in[d],k\in[n_i]}$, let us denote by $\mathcal{L}_d^{k_{ij},\mathrm{n}}(\mathbf{c})$ the set of *labeled* forests in which the vertex of type i with label k has $c_{i,j,k}$ offspring of type j.

$\mathbf{c}$ is called the *indegree tuple* of the forest $\mathbf{f} \in \mathcal{L}_d^{k_{ij},\mathrm{n}}(\mathbf{c})$.

Theorem (Bernardi, Morales '14)

For any indegree tuple $\mathbf{c} = (c_{i,j,k})_{i,j\in[d],k\in[n_i]}$, the number of forests with $\mathbf{c}$ as the indegree tuple is

$$\left|\mathcal{L}_d^{k_{ij},\mathrm{n}}(\mathbf{c})\right| = \frac{\prod_{j=1}^{d}(n_j-1)!}{\prod_{i\in[d]} r_i! \prod_{i,j\in[d],k\in[n_i]} c_{i,j,k}!} \det(-k_{ij}).$$