Infinitely ramified point measures and branching Lévy processes

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Joint work with Jean Bertoin (Universität Zürich)

Universität Zürich

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Outline

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Infinitely divisible random variable

Definition

A random variable X is said to be infinitely divisible if for any $n \in \mathbb{N}$, there exists a random walk $S^{(n)}$ such that $X = S_n^{(n)}$ in law.

Theorem (Lévy, Khintchine)

There exists a unique Lévy process $(X_t, t \geq 0)$ such that $X_1 = X$, characterized by (σ^2, a, π), satisfying

$$\sigma^2 \geq 0, \quad a \in \mathbb{R}, \quad \int_{\mathbb{R}} (1 \wedge x^2) \pi(dx) < +\infty.$$

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Representation of a Lévy process

Property

$$\mathbf{E}\left(e^{i\xi X_t}\right) = \exp\left[t\left(-\frac{\sigma^2}{2}\xi^2 + ia\xi + \int_{\mathbb{R}} e^{i\xi x} - 1 - i\xi x \mathbf{1}_{\{|x|<1\}}\pi(dx)\right)\right].$$

$$\textit{Moreover, if } \int_1^{+\infty} e^{\theta x}\pi(dx) < +\infty, \ \textit{then } \mathbf{E}(e^{\theta X_1}) < +\infty.$$

Representation of a Lévy process

Theorem (Lévy-Itō decomposition)

A Lévy process X with characteristics (σ^2, a, π) can be described as

$$X_t = \sigma B_t + at + \int_0^t \int_{|x|\geq 1} xN(dsdx) + \lim_{\varepsilon \to 0} \int_0^t \int_{\varepsilon<|x|<1} xN(dsdx) - t \int_{\varepsilon<|x|<1} x\pi(dx),$$

where B is a Brownian motion and N a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $dt\pi(dx)$.

Locally finite point measures

Definition

We denote by $\mathcal{P}$ the set of point Radon measures on $\mathbb{R}$ giving finite mass to $\mathbb{R}^+$, which is identified to the set of nonincreasing sequences converging toward $-\infty$ by

$$\mu \in \mathcal{P} \iff \mu = \sum_{j=1}^{+\infty} \delta_{x_j}, \text{ avec } x_1 \geq x_2 \geq \ldots \text{ et } \lim_{j \to +\infty} x_j = -\infty.$$

Notation

For all $x \in \mathbb{R}$ and $\mu \in \mathcal{P}$, we set $\tau_x \mu = \sum_{j \in \mathbb{N}} \delta_{x+x_j}$ the translation operator.

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Branching random walk

Definition

Let $(\mu_{n,j}, n, j \in \mathbb{N})$ be a family of i.i.d. random variables in $\mathcal{P}$. We assume there exists $\theta \geq 0$ satisfying

$$\mathbf{E}\left(\int_{\mathbb{R}} e^{\theta x}\mu_{1,1}(dx)\right) < +\infty.$$

A branching random walk with reproduction law $\mu_{1,1}$ is a process $(Z_n, n \in \mathbb{N})$ taking values in $\mathcal{P}$ which can be defined as follows:

$$Z_0 = \delta_0 \quad \text{and} \quad Z_{n+1} = \sum_{j\in\mathbb{N}} \tau_{z_{n,j}}\mu_{(n+1),j},$$

where $(z_{n,j}, j \in \mathbb{N})$ is the sequence of atoms associated to Z_n.

A graphic representation

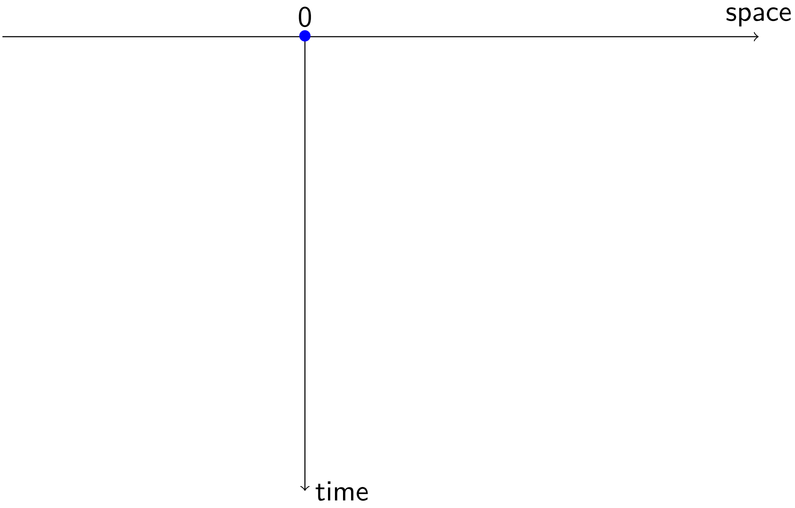
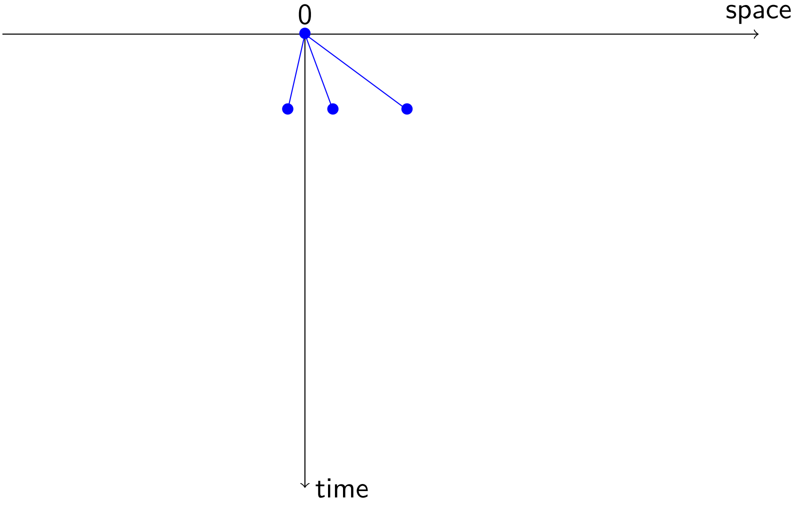


Figure: A branching random walk on $\mathbb{R}$

A graphic representation



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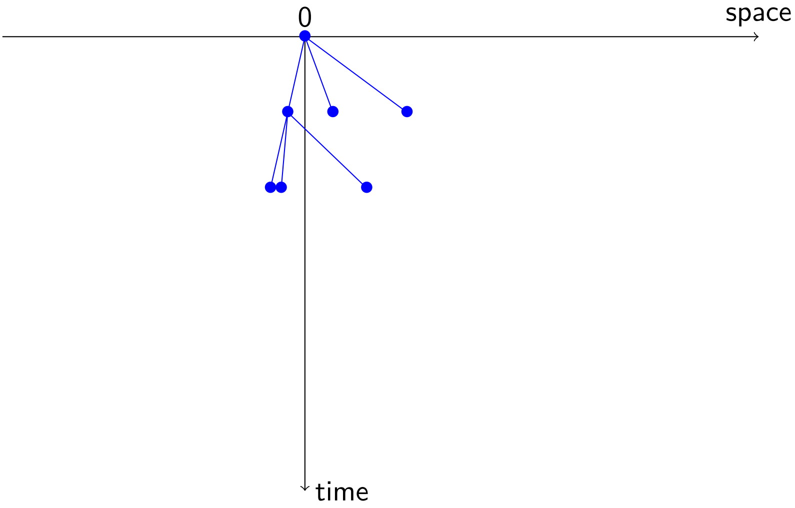

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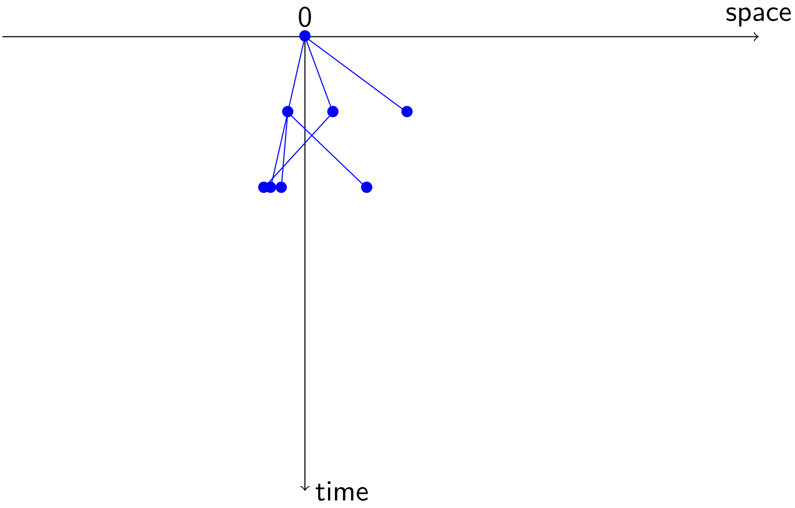

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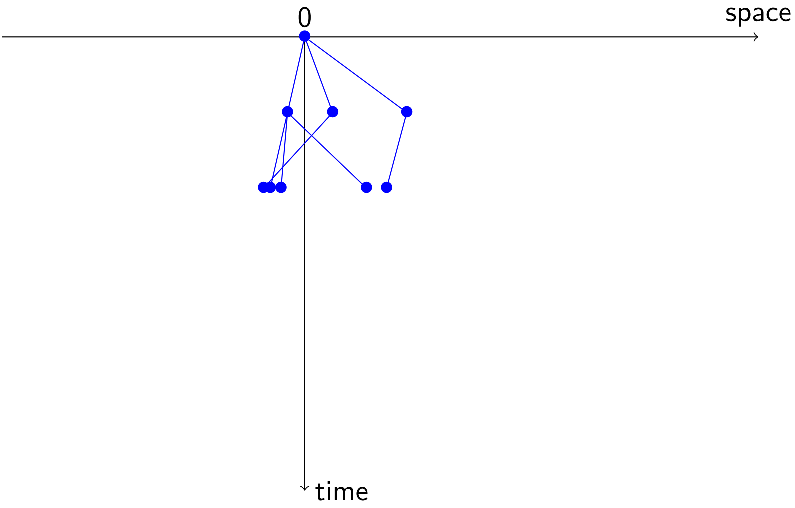

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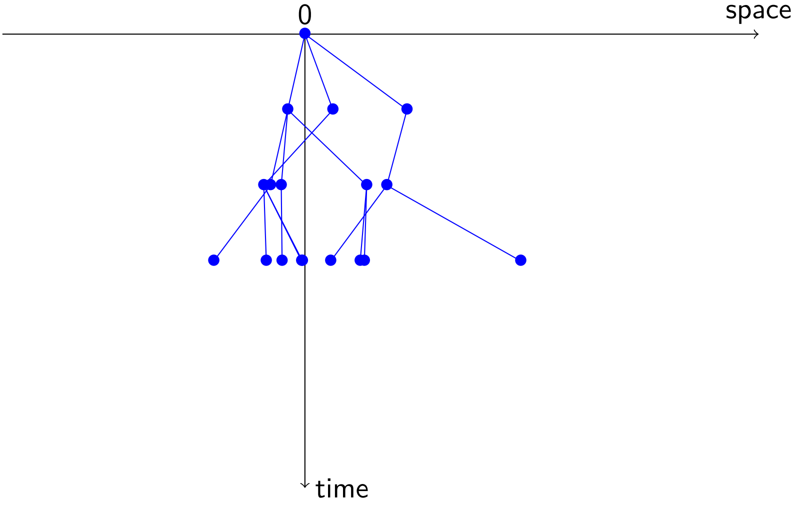

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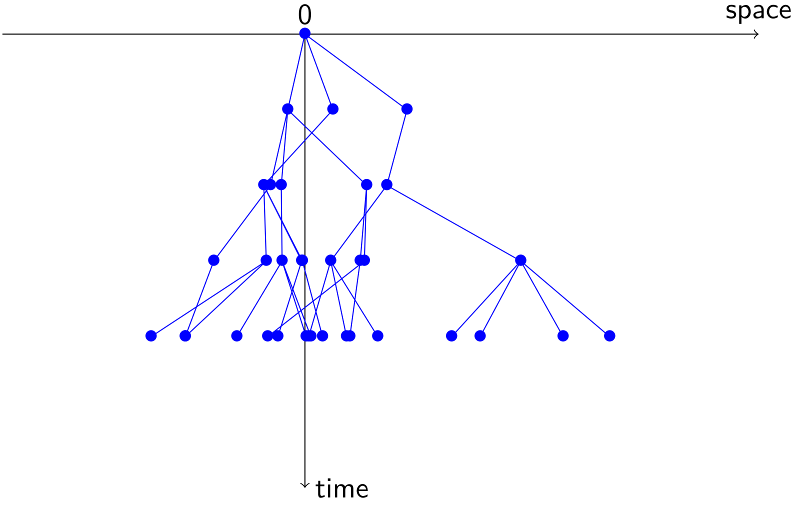

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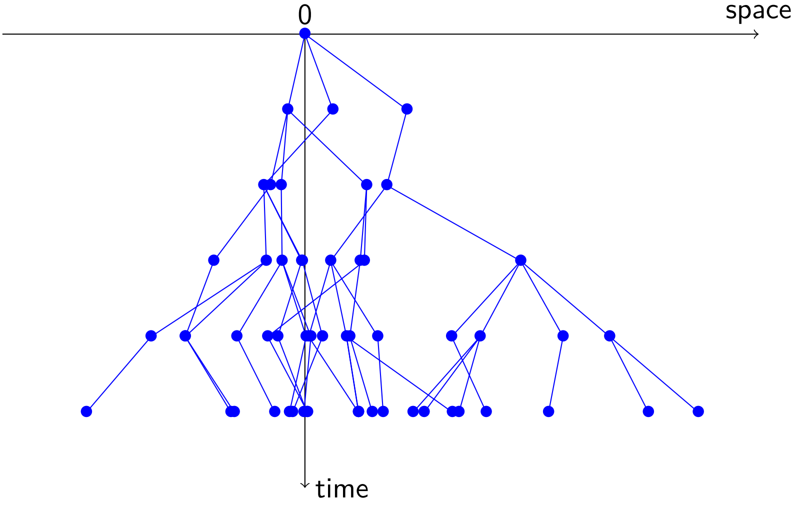

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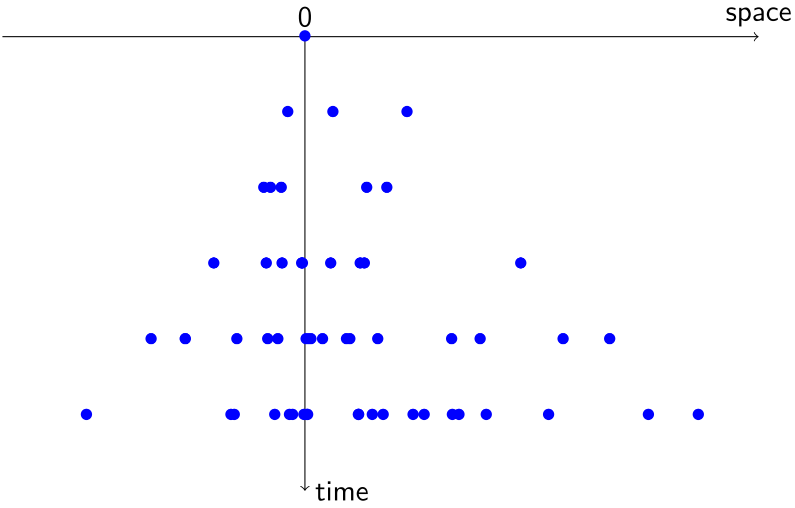

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Infinitely ramified point measure

Definition

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Theorem (Lévy, Khintchine)

There exists a unique Lévy process $(X_t, t \geq 0)$ *such that* $X_1 = X$*, characterized by* (σ^2, a, π)*, satisfying*

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Theorem (Bertoin-M.)

If there exists $\theta \geq 0$ such that $\mathbf{E}(\int e^{\theta x} \mathcal{Z}(dx)) < +\infty$, there exists a branching Lévy process $(Z_t, t \in \mathbb{R}_+)$ such that $Z_1 = Z$, characterized by (σ^2, a, Λ), satisfying

$$\sigma^2 \geq 0, \quad a \in \mathbb{R}, \quad \int (1 \wedge x_1^2) + \sum_{k=1}^{+\infty} e^{\theta x_k} - 1 - \theta x_1 \mathbf{1}_{\{|x_1| \leq 1\}} \Lambda(d\mathbf{x}) < +\infty.$$

Representation of an infinitely ramified point measure

Proposition (Uchiyama process)

If $\int_{\mathcal{P}}(\mu(\mathbb{R}) - 1)d\Lambda < +\infty$, *a branching Lévy process evolves as follows*

- *each particle moves according to an independent Lévy process;*
- *each particle gives independently birth to children around its poisition at exponential rate.*

We say that such a process has finite birth intensity.

Theorem

A general branching Lévy process is obtained as the increasing limit of finite birht intensity branching Lévy processes.

Representation of an infinitely ramified point measure

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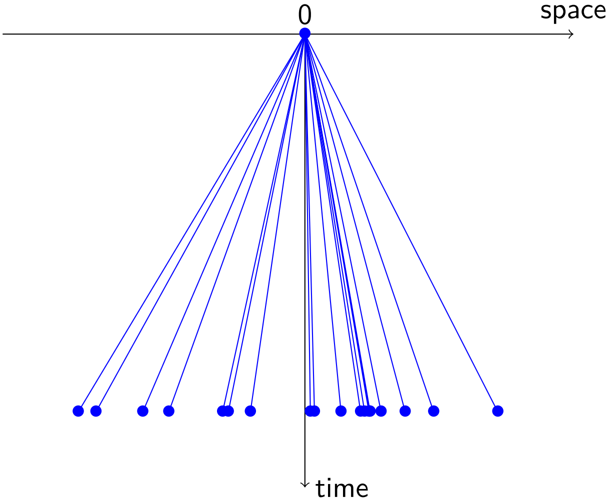

Figure: Some decompositions of an infinitely ramified point measure

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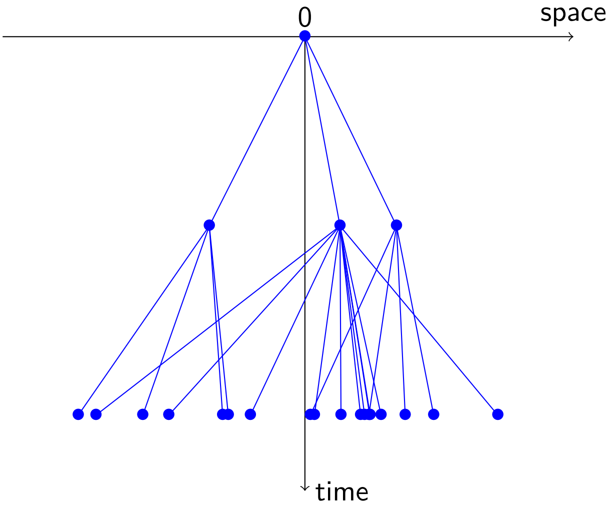

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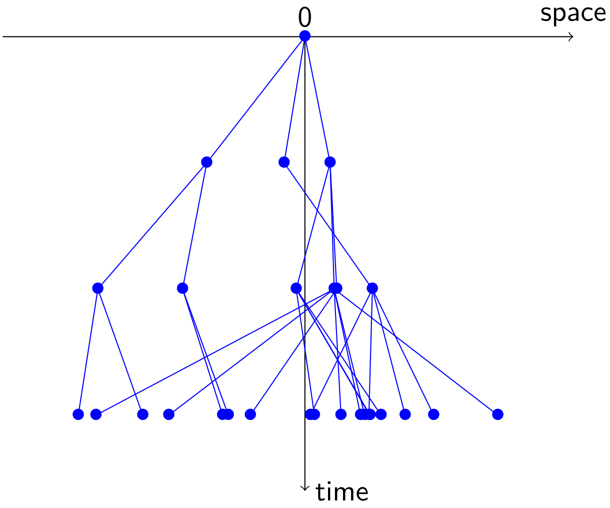

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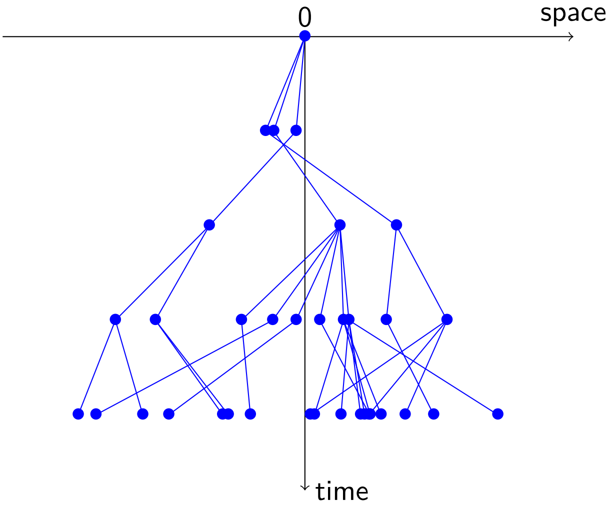

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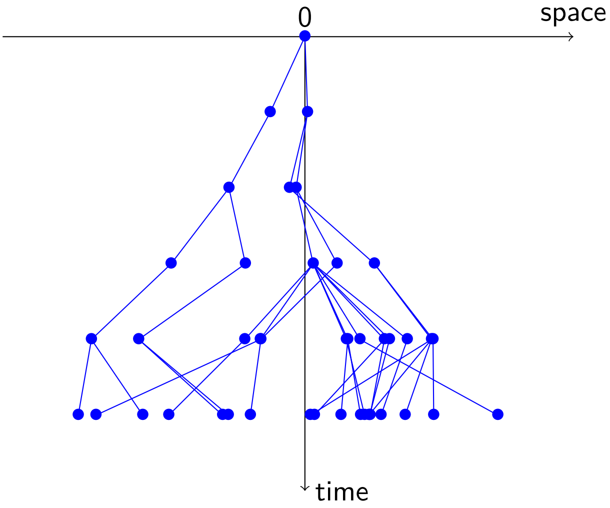

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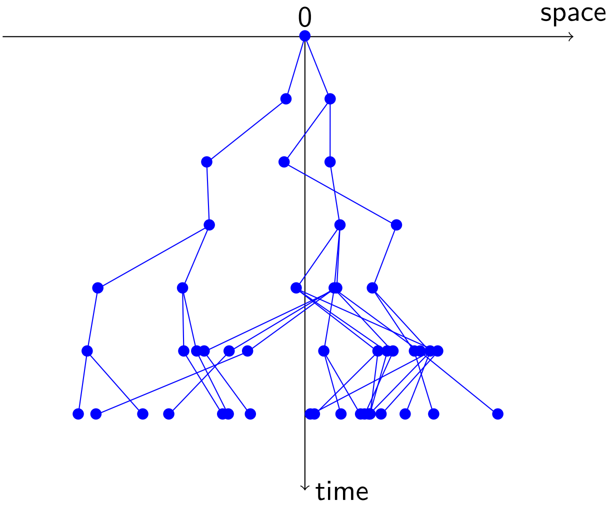

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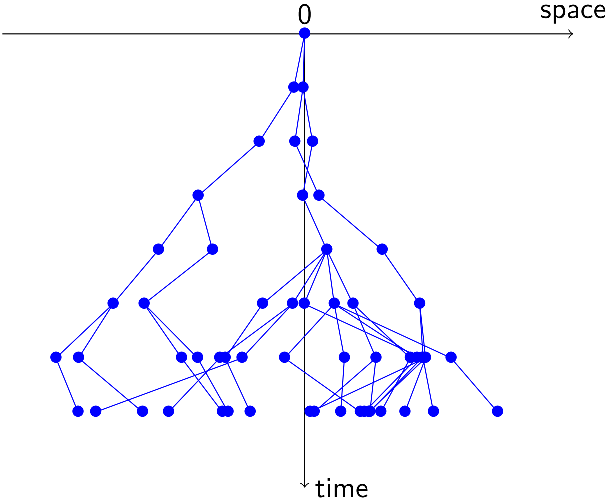

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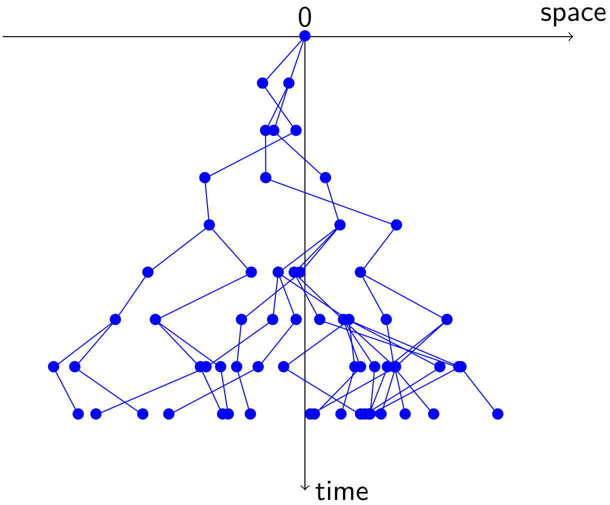

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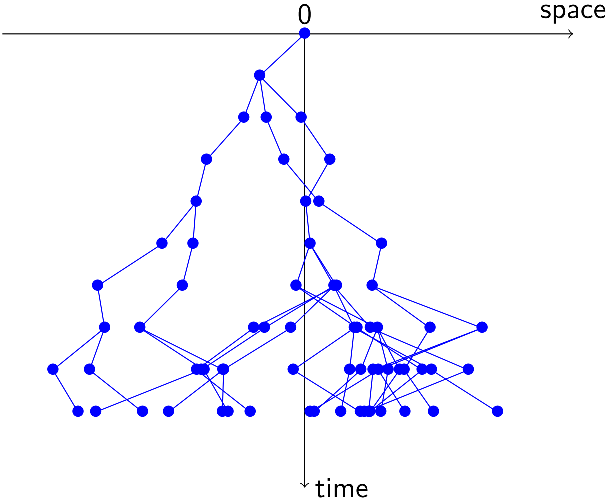

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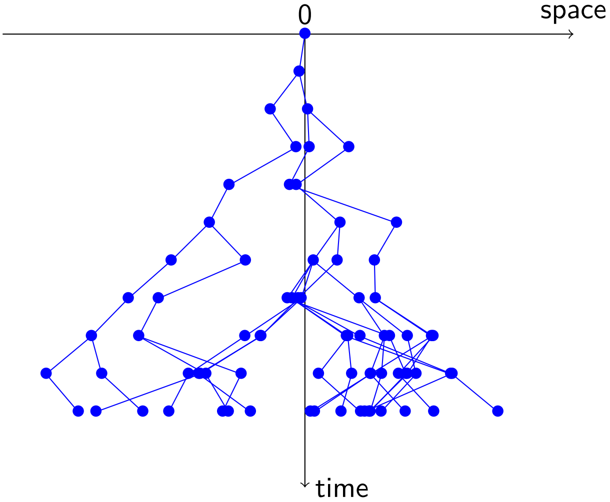

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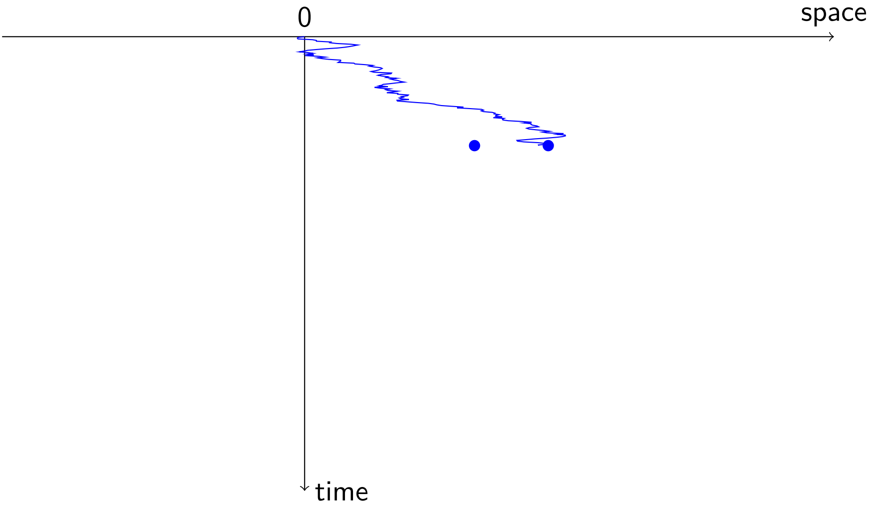
Branching Lévy processes : finite birth intensity case



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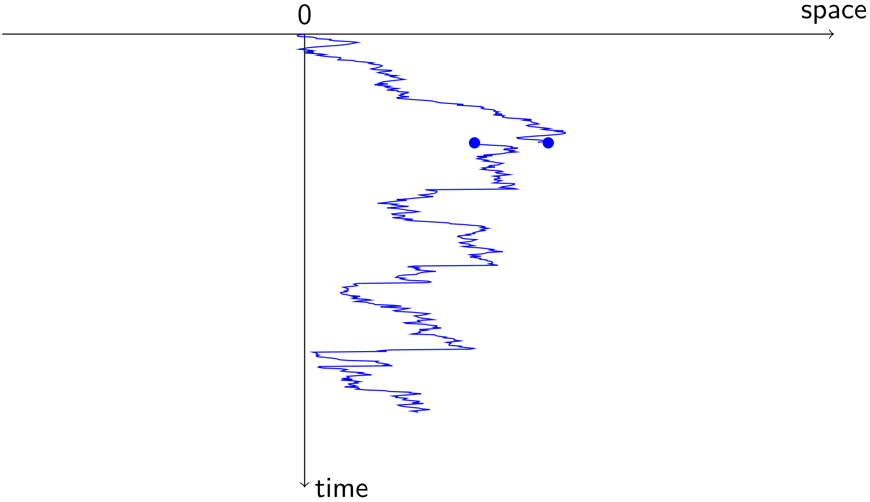

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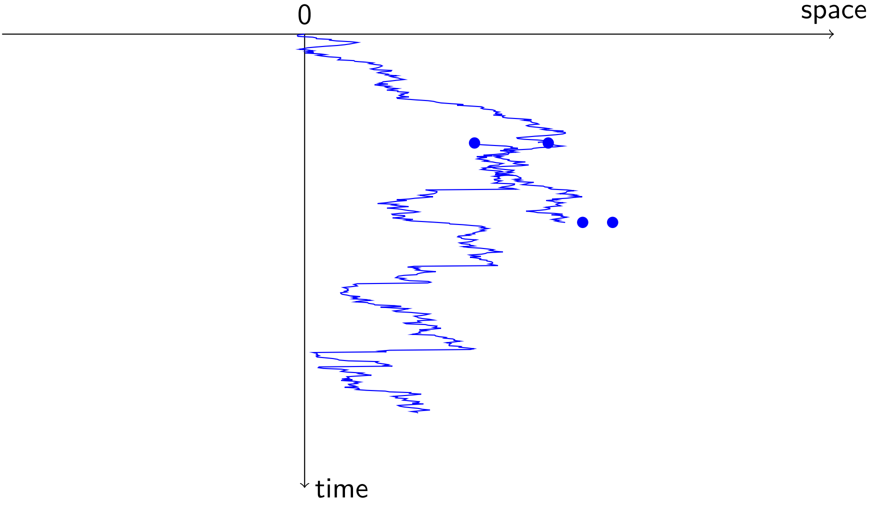

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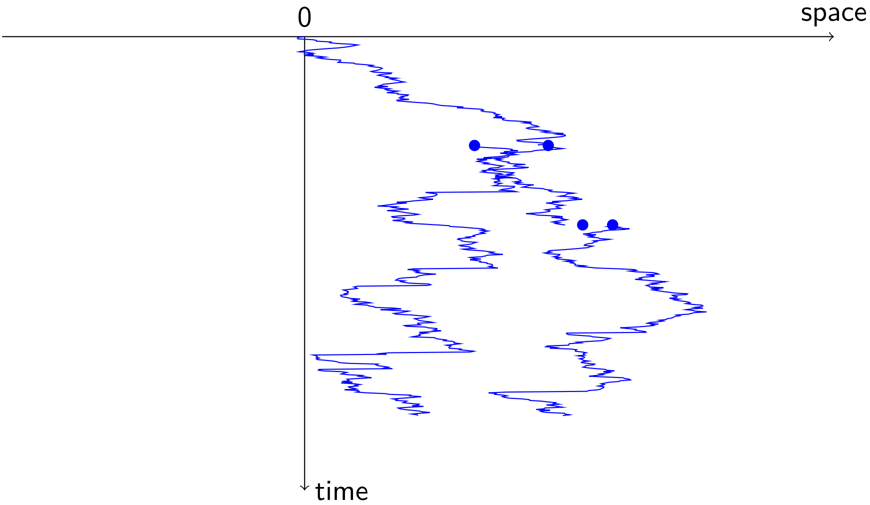

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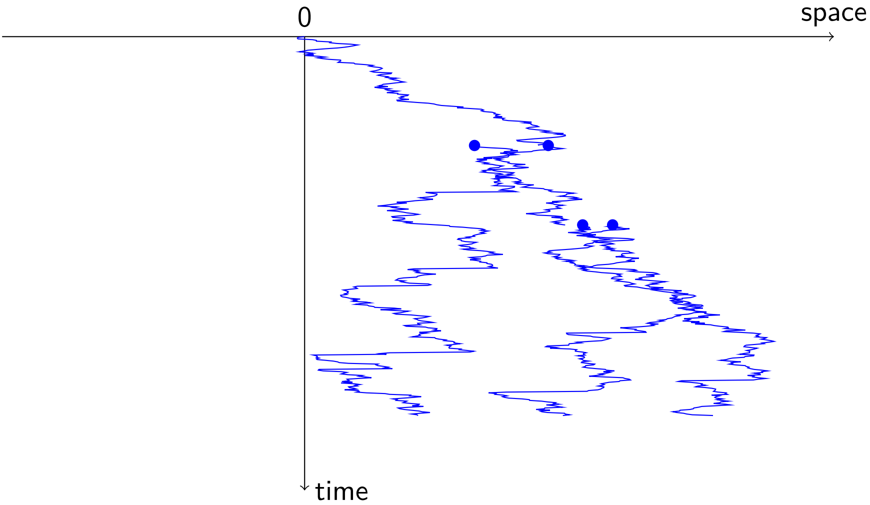

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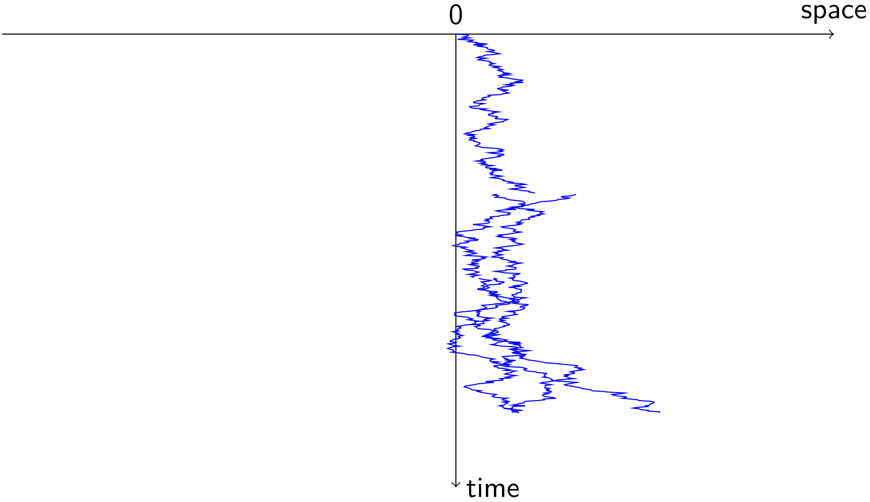

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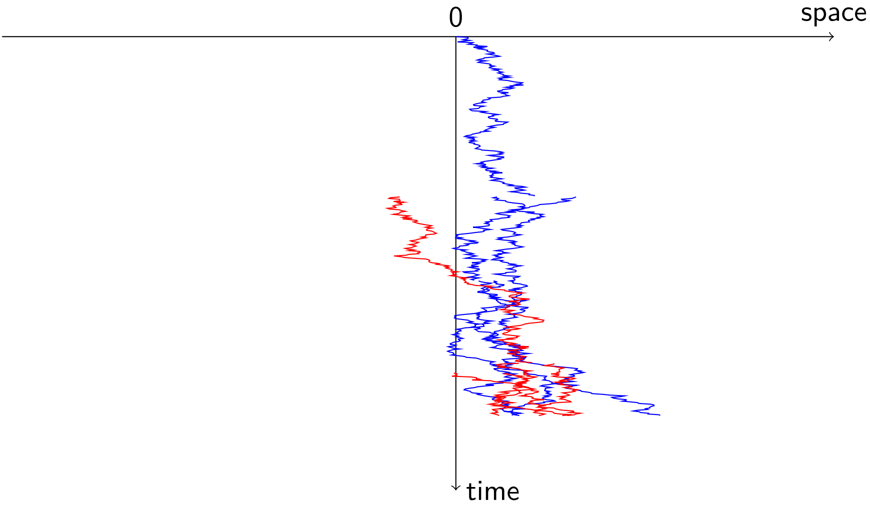

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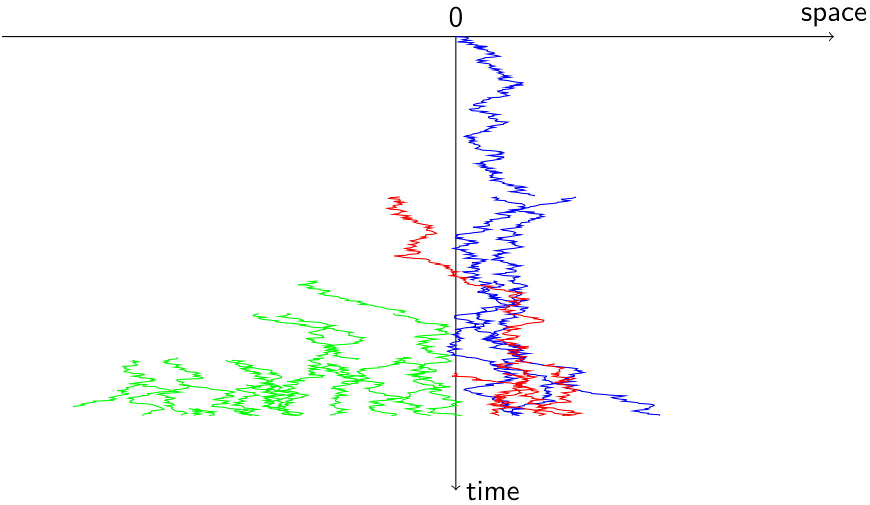

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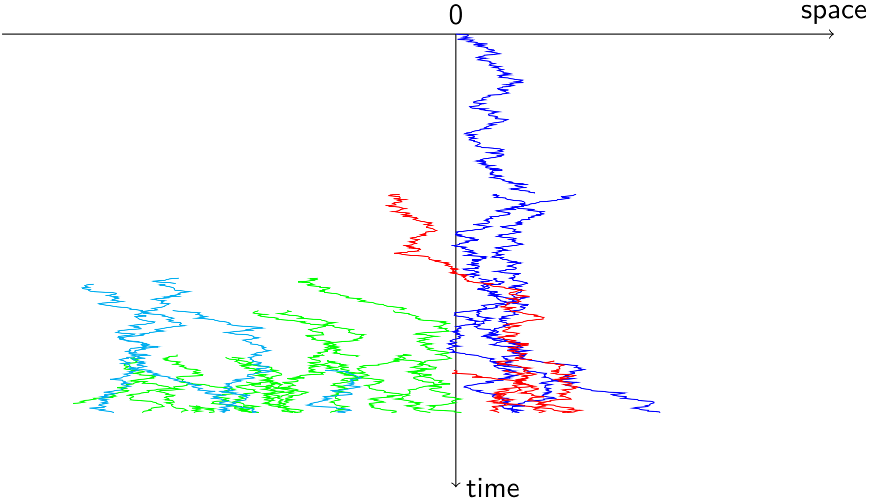

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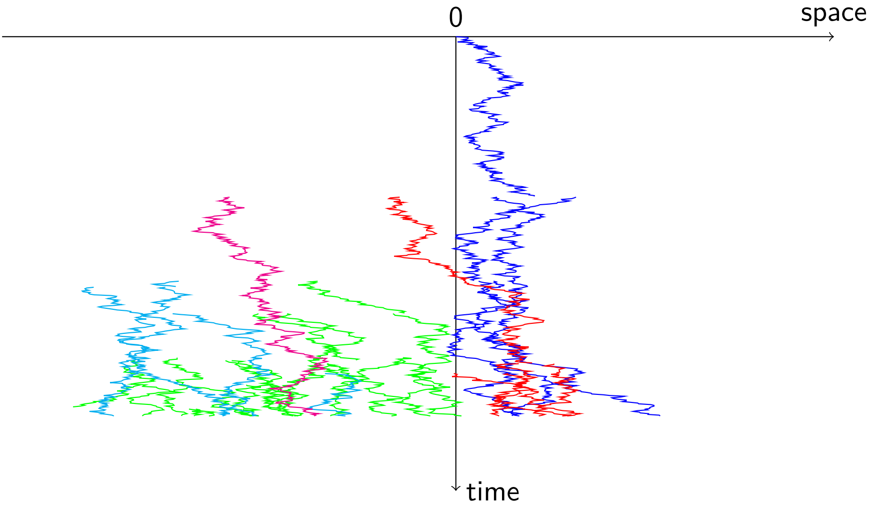

Figure: Construction of a branching Lévy process

Outline

Branching convolution operator

Definition

Let P and Q be the laws of two point measures. We denote by $P \circledast Q$ the law obtained by making one step of branching random walk with law P, followed by one step of a branching random walk with law Q.

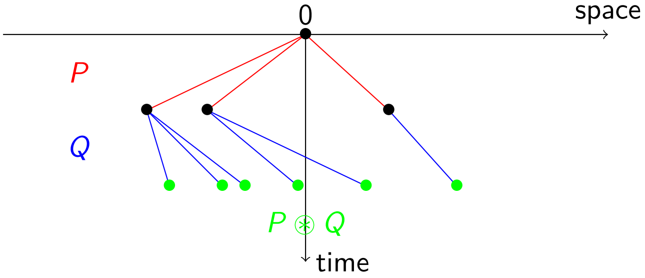

Figure: Branching convolution operator

Construction of nested branching random walks I

Lemma

Let P be the law of an infinitely ramified point measure. For all $n \in \mathbb{N}$, the set

$$R_n(P) = \left\{ Q : Q^{\circledast 2^n} = P \right\}$$

is a non-empty compact set on which $\circledast$ is continuous.

Corollary

There exists a sequence (Q_n) such that $Q_0 = P$ and $Q_{n+1} \circledast Q_{n+1} = Q_n$.

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Construction of nested branching random walks II

Definition

A nested branching random walk is a $\mathcal{P}$-valued process $(Z_t, t \in D)$ such that for all $n \in \mathbb{N}$ $(Z_{k2^{-n}}, k \in \mathbb{N})$ is a branching random walk.

Theorem

If $\mathcal{Z}$ is an infinitely ramified point measure, there exists a nested branching random walk Z such taht $\mathcal{Z} = Z_1$ in law.

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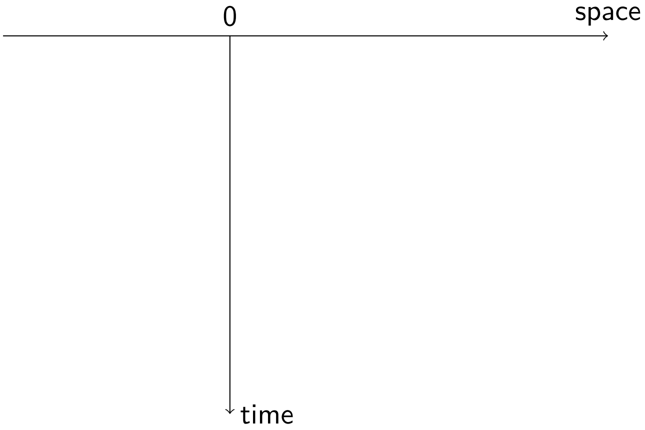

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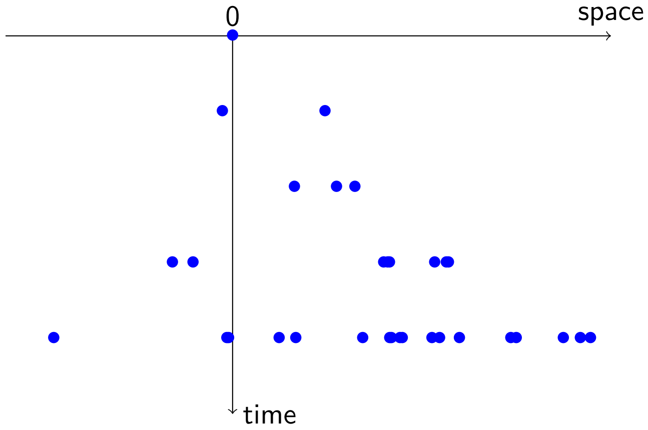

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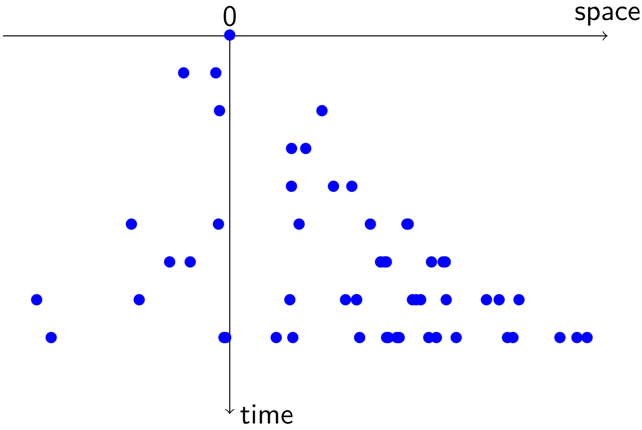

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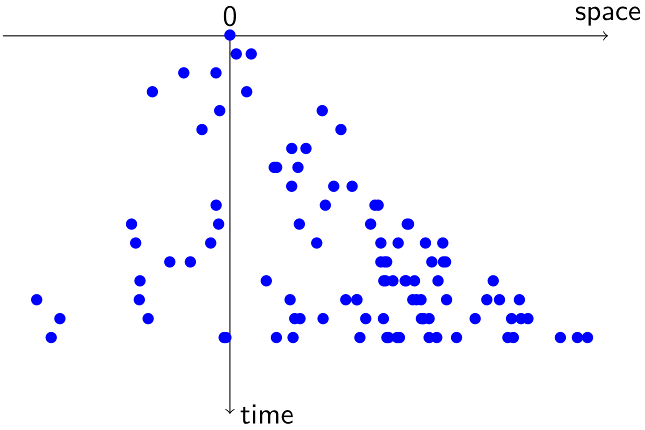

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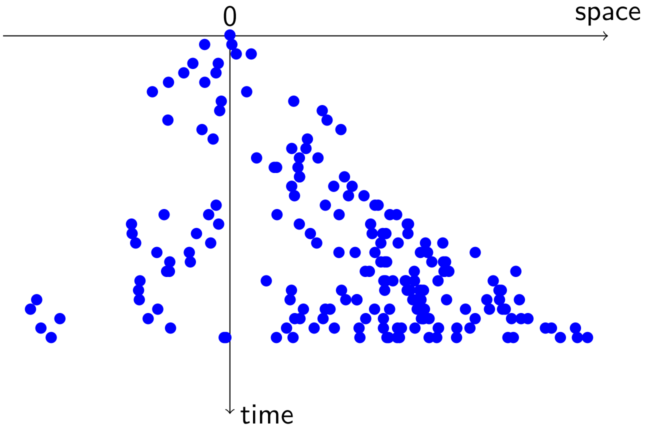

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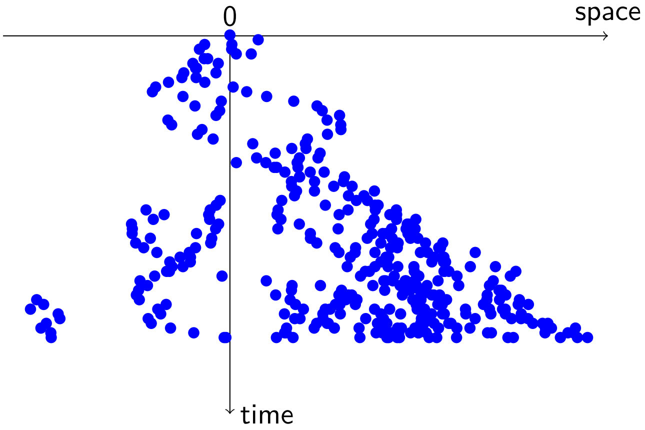

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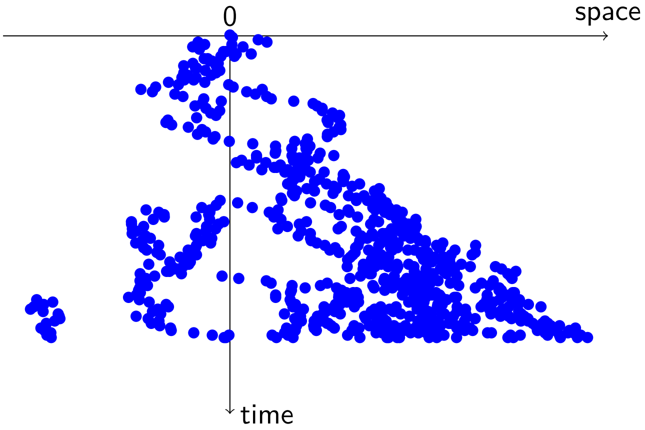

Many-to-one lemma

Branching random walk

Let $(Z_n, n \in \mathbb{N})$ be a branching random walk. We set

$$\kappa(\theta) = \log \mathbf{E}\left(\int e^{\theta x} Z_1(dx) \right).$$

Lemma

There exists a random walk S such that for all $n \in \mathbb{N}$ and for all measurable bounded function f,

$$\mathbf{E}\left(\int f(x) Z_n(dx) \right) = \mathbf{E}\left(e^{-\theta S_n + n\kappa(\theta)} f(S_n) \right).$$

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Moreover, for all $\lambda \in \mathbb{R}$, $\mathbf{E}\left(e^{i\lambda \xi_1} \right) = e^{-\kappa(\theta)} \mathbf{E}\left(\int e^{(\theta + i\lambda)x} Z_1(dx) \right)$.

Càdlàg extension

Proposition

For all $\lambda \in \mathbb{R}$ and $t \geq 0$, we set

$$M_t(\theta) = e^{-t\kappa(\theta+i\lambda)} \int e^{(\theta+i\lambda)} Z_t(dx).$$

Note that $((M_t(\theta), \theta \in \mathbb{R}), t \in D)$ is a martingale in the space of continuous function, endowed with the uniform convergence on compact sets. It therefore admits a càdlàg extension.

Corollary

As a result, $(Z_t, t \in D)$ has a càdlàg extension $(Z_t, t \geq 0)$.

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Many-to-one lemma

Let ξ be the Lévy process such that for all $\lambda \in \mathbb{R}$,

$$\mathbf{E}\left(e^{i\lambda\xi_1}\right) = e^{-\kappa(\theta)}\mathbf{E}\left(\int e^{(\theta+i\lambda)x} Z_1(dx)\right)$$

Lemma

For all $t \geq 0$ and measurable bounded function f, we have

$$\mathbf{E}\left(\int f(x) Z_t(dx)\right) = \mathbf{E}\left(e^{-\theta\xi_t + t\kappa(\theta)} f(\xi_t)\right).$$

Strong branching property

Theorem

Let T be a stopping time of $(Z_t, t \geq 0)$. We have the strong branching property:

$$(Z_{T+s}, s \geq 0) = \left(\sum_{j \in \mathbb{N}} \tau_{z_T^j} Z_s^j, s \geq 0 \right) \text{ in law,}$$

with $(Z_s^j, s \geq 0)$ i.i.d. copies of Z, and (z_T^j) the sequence of atoms associated to Z_T.

Outline

The total mass process

Let Z be a nested branching random walk such that $\mathbf{E}(Z_1(\mathbb{R})) < +\infty$.

Lemma

The process $(Z_t(\mathbb{R}), t \geq 0)$ is a continuous time Galton-Watson process.

Proposition

Let $T = \inf\{t \geq 0 : Z_t(\mathbb{R}) \neq 1\}$, for $s \leq T$ ζ_s defined by $Z_s = \delta_{\zeta_s}$, and $\Delta = \tau_{-\zeta_{T-}} Z_T$. We have

1. *T has an exponential distribution;*
2. *$(\zeta_s, s \leq T)$ is an independent Lévy process, killed at time T ;*
3. *Δ is an independent point measure, with $\mathbf{E}(\Delta(\mathbb{R})) < +\infty$.*

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Characteristics of the branching Lévy process

Notation

We set

1. λ the parameter of T;
2. (σ^2, a, π) the characteristics of ζ;
3. ρ the law of Δ.

Theorem

The process $(Z_t, t \geq 0)$ is a branching Lévy process with characteristics (σ^2, a, Λ), where

$$\Lambda(d\mathbf{x}) = \pi(dx_1)\mathbf{1}_{\{x_2=-\infty\}} + \lambda\rho(d\mathbf{x}).$$

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Outline

Introduction of a" natural" genealogy

Definition

A family of partitions of $\mathbb{N}$ $(\Pi_{s,t}, s \leq t \in D)$ is a genealogy if for all r, s, t:

$$\Pi_{r,t} = \mathrm{Coag}\left(\Pi_{s,t}, \Pi_{r,s}\right).$$

Proposition

Let Z be a nested branching random walk, there exists a genealogy associated to Z such that for all $n \in \mathbb{N}$, $(Z_{k2^{-n}}, k \geq 0)$ has a genealogy compatible with Π.

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Uniqueness of the natural genealogy ?

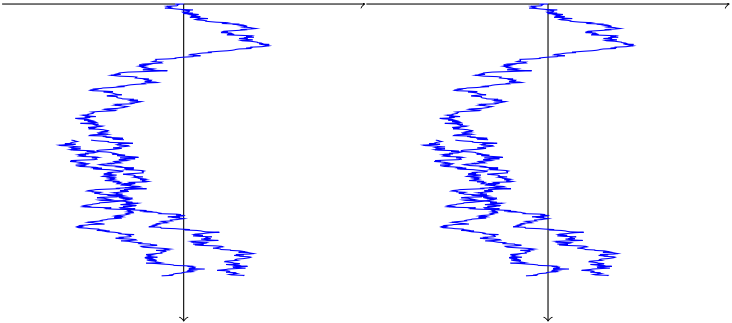

Figure: Multiple genealogical reconstructions can be possible

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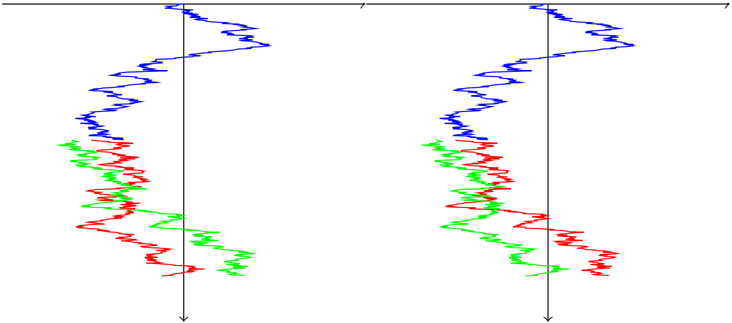

Figure: Multiple genealogical reconstructions can be possible

Trajectorial "Many-to-one" lemma

Lemma

Thanks to the genealogy associated to Z, we can define for all atom $z_{t,j}$ of Z_t the trajectory (on D) followed by this atom. We have

$$\mathbf{E}\left(\sum_{j\in\mathbb{N}} f(z_{t,j}(s), s \leq t)\right) = \mathbf{E}\left(e^{-\theta\xi_t + t\kappa(\theta)} f(\xi_s, s \leq t)\right).$$

In particular, every trajectory admits a unique càdlàg extension to $\mathbb{R}$.

The censored process

We can define, for all $n \in \mathbb{N}$ the process

$$Z_t^{(n)} = \sum_{j \in \mathbb{N}} \delta_{z_{t,j}} \mathbf{1}_{\left\{\min_{s \leq t} \Delta z_{j,t}(s) \geq -n\right\}}.$$

This process remains a nested branching random walk. Moreover, we have

$$\mathbf{E}(Z_t^{(n)}(\mathbb{R})) = \mathbf{E}\left(e^{-\theta\xi_t + t\kappa(\theta)} \mathbf{1}_{\left\{\min_{s \leq t} \Delta \xi_s \geq -n\right\}}\right) < +\infty.$$

Thus we can use the finite intensity case to conclude.