Quantum Bessel processes and birth and death processes on partitions

Marcin Świeca

Probability and Analysis
15.05.2017 - 19.05.2017 Będlewo

Based on :

1. W.Matysiak, M. Ś, *Zonal polynomials and a multidimensional quantum Bessel process*, Stochastic Process. Appl., 2015.
2. W.Matysiak, M. Ś, *Jordan algebras and quantum Bessel processes*, Int. Math. Res. Not. IMRN, 2016.

Biane's quantum Bessel process

Philippe Biane (1996) - a construction of an analogue of the Bessel process (*quantum Bessel process*): a Markov process (in a classical, not non-commutative sense) living on a subset of $\mathbb{R}^d$.

Ingredients:

- the Heisenberg group: $H = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, w)(z', w') = (z+z', w+w'+\text{Im}\,(z'|z)),\ z, z' \in \mathbb{C}^n,\ w, w' \in \mathbb{R},$$

- the function $\psi(z, w) = iw - \frac{1}{2}||z||^2$ on H with the property: $\forall\ t \geq 0,\ H \ni g \mapsto \exp[t\psi(g)]$ is positive definite.

It can be shown that $Q_t : L^1(H) \to L^1(H)$

$$Q_t f(g) = \exp[t\psi(g)] f(g)$$

extends to a semigroup of (completely) positive contractions on $\mathrm{C}^*(H)$.

Biane's quantum Bessel process

Philippe Biane (1996) - a construction of an analogue of the Bessel process (*quantum Bessel process*): a Markov process (in a classical, not non-commutative sense) living on a subset of $\mathbb{R}^d$.

Ingredients:

- the Heisenberg group: $H = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, w)(z', w') = (z+z', w+w'+\mathrm{Im}\,(z'|z)),\ z, z' \in \mathbb{C}^n,\ w, w' \in \mathbb{R},$$

- the function $\psi(z, w) = iw - \frac{1}{2}||z||^2$ on H with the property: $\forall\ t \geq 0,\ H \ni g \mapsto \exp[t\psi(g)]$ is positive definite.

It can be shown that $Q_t : L^1(H) \to L^1(H)$

$$Q_t f(g) = \exp[t\psi(g)] f(g)$$

extends to a semigroup of (completely) positive contractions on $\mathrm{C}^*(H)$.

Biane's quantum Bessel process

Philippe Biane (1996) - a construction of an analogue of the Bessel process (*quantum Bessel process*): a Markov process (in a classical, not non-commutative sense) living on a subset of $\mathbb{R}^d$.
Ingredients:

- the Heisenberg group: $H = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, w)(z', w') = (z+z', w+w'+\mathrm{Im}\,(z'|z)),\ z, z' \in \mathbb{C}^n,\ w, w' \in \mathbb{R},$$

- the function $\psi(z, w) = iw - \frac{1}{2}||z||^2$ on H with the property: $\forall\ t \geq 0,\ H \ni g \mapsto \exp[t\psi(g)]$ is positive definite.

It can be shown that $Q_t : L^1(H) \to L^1(H)$

$$Q_t f(g) = \exp[t\psi(g)]f(g)$$

extends to a semigroup of (completely) positive contractions on $C^*(H)$.

Biane's quantum Bessel process

Philippe Biane (1996) - a construction of an analogue of the Bessel process (*quantum Bessel process*): a Markov process (in a classical, not non-commutative sense) living on a subset of $\mathbb{R}^d$.
Ingredients:

- the Heisenberg group: $H = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, w)(z', w') = (z+z', w+w'+\text{Im}\,(z'|z)),\ z, z' \in \mathbb{C}^n,\ w, w' \in \mathbb{R},$$

- the function $\psi(z, w) = iw - \frac{1}{2}||z||^2$ on H with the property: $\forall\ t \geq 0,\ H \ni g \mapsto \exp[t\psi(g)]$ is positive definite.

It can be shown that $Q_t : L^1(H) \to L^1(H)$

$$Q_t f(g) = \exp[t\psi(g)] f(g)$$

extends to a semigroup of (completely) positive contractions on $C^*(H)$.

Biane's quantum Bessel process

Philippe Biane (1996) - a construction of an analogue of the Bessel process (*quantum Bessel process*): a Markov process (in a classical, not non-commutative sense) living on a subset of $\mathbb{R}^d$.
Ingredients:

- the Heisenberg group: $H = \mathbb{C}^n \times \mathbb{R}$ with the group law

  $$(z, w)(z', w') = (z+z', w+w'+\mathrm{Im}\,(z'|z)),\ z, z' \in \mathbb{C}^n,\ w, w' \in \mathbb{R},$$

- the function $\psi(z, w) = iw - \frac{1}{2}||z||^2$ on H with the property: $\forall\ t \geq 0,\ H \ni g \mapsto \exp[t\psi(g)]$ is positive definite.

It can be shown that $Q_t : L^1(H) \to L^1(H)$

$$Q_t f(g) = \exp[t\psi(g)]f(g)$$

extends to a semigroup of (completely) positive contractions on $\mathrm{C}^*(H)$.

Biane's quantum Bessel process

Philippe Biane (1996) - a construction of an analogue of the Bessel process (*quantum Bessel process*): a Markov process (in a classical, not non-commutative sense) living on a subset of $\mathbb{R}^d$.

Ingredients:

- the Heisenberg group: $H = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z, w)(z', w') = (z+z', w+w'+\mathrm{Im}\,(z'|z)),\ z, z' \in \mathbb{C}^n,\ w, w' \in \mathbb{R},$$

- the function $\psi(z, w) = iw - \frac{1}{2}||z||^2$ on H with the property: $\forall\ t \geq 0,\ H \ni g \mapsto \exp[t\psi(g)]$ is positive definite.

It can be shown that $Q_t : L^1(H) \to L^1(H)$

$$Q_t f(g) = \exp[t\psi(g)]f(g)$$

extends to a semigroup of (completely) positive contractions on $\mathrm{C}^*(H)$.

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.

How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0(\sigma(\mathcal{B}))$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.
How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0(\sigma(\mathcal{B}))$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.
How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0(\sigma(\mathcal{B}))$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.
How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0(\sigma(\mathcal{B}))$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.
How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0(\sigma(\mathcal{B}))$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.
How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0\left(\sigma(\mathcal{B})\right)$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

Classical Markov processes from non-commutative ones

H - non-abelian, so $\mathrm{C}^*(H)$ - non-commutative and $(Q_t)_t$ on $\mathrm{C}^*(H)$ is the semigroup of non-commutative Brownian motion.
How can one construct some classical Markov processes in such setting? If

- $\mathcal{A}$ - a non-commutative C^*-algebra,
- $(\Phi_t)_t$ – a semigroup of positive contractions on $\mathcal{A}$,
- $\mathcal{B}$ – a commutative sub-C^*-algebra of $\mathcal{A}$.

Then:

- Gelfand theorem $\Rightarrow \mathcal{B} \cong C_0\left(\sigma(\mathcal{B})\right)$, where $\sigma(\mathcal{B})$ – the Gelfand spectrum of $\mathcal{B}$;
- if $\mathcal{B}$ is $(\Phi_t)_t$-invariant, then the restriction of $(\Phi_t)_t$ to $\mathcal{B}$ defines a Markov semigroup on $\sigma(\mathcal{B})$ (via the Riesz representation theorem for C_0)

A commutative sub-C^*-algebra of $C^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions ($f(z,w)=F(|z|,w)$) is commutative – equivalently, $(U(n)\ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(C^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f\mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n)\ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w)=\binom{m+n-1}{m}^{-1}\exp\left[i\tau w-\frac{|\tau||z|^2}{2}\right]L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau\neq 0$, $m\in\mathbb{N}$, or

$$\phi_\mu(z,w)=\Gamma(n)J_{n-1}(\mu|z|)\left(\frac{\mu|z|}{2}\right)^{1-n},\quad \mu\geq 0,$$

A commutative sub-C^*-algebra of $C^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions $(f(z,w) = F(|z|,w))$ is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(C^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})dg,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0$, $m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n) J_{n-1}(\mu|z|) \left(\frac{\mu|z|}{2}\right)^{1-n}, \quad \mu \geq 0.$$

A commutative sub-C^*-algebra of $C^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions ($f(z,w) = F(|z|,w)$) is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(C^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0$, $m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n) J_{n-1}(\mu|z|) \left(\frac{\mu|z|}{2}\right)^{1-n}, \quad \mu \geq 0.$$

A commutative sub-C^*-algebra of $\mathrm{C}^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions ($f(z,w) = F(|z|,w)$) is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(\mathrm{C}^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0,\ m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n) J_{n-1}(\mu|z|) \left(\frac{\mu|z|}{2}\right)^{1-n}, \quad \mu \geq 0,$$

A commutative sub-C^*-algebra of $\mathrm{C}^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions ($f(z,w) = F(|z|,w)$) is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(\mathrm{C}^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0$, $m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n)J_{n-1}(\mu|z|)\left(\frac{\mu|z|}{2}\right)^{1-n}, \quad \mu \geq 0,$$

A commutative sub-C^*-algebra of $\mathrm{C}^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions $(f(z,w) = F(|z|,w))$ is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(\mathrm{C}^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0,\ m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n) J_{n-1}(\mu|z|)\left(\frac{\mu|z|}{2}\right)^{1-n}, \quad \mu \geq 0,$$

A commutative sub-C^*-algebra of $\mathrm{C}^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions ($f(z,w) = F(|z|,w)$) is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(\mathrm{C}^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0$, $m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n) J_{n-1}(\mu|z|) \left(\frac{\mu|z|}{2}\right)^{1-n}, \quad \mu \geq 0,$$

A commutative sub-C^*-algebra of $\mathrm{C}^*(H)$

Get back to the Biane's construction:

- the convolution subalgebra $L^1(H/U(n))$ of *radial* functions $(f(z,w) = F(|z|,w))$ is commutative - equivalently, $(U(n) \ltimes H, U(n))$ is a *Gelfand pair*,
- the Gelfand spectrum $\sigma(\mathrm{C}^*(H/U(n)))$ is explicitly known: all (non-trivial) complex homomorphisms are of the form

$$f \mapsto \int_H f(g)\phi(g^{-1})\mathrm{d}g,$$

where functions ϕ (the *spherical functions* of the Gelfand pair $(U(n) \ltimes H, U(n))$) are

$$\phi_{\tau,m}(z,w) = \binom{m+n-1}{m}^{-1} \exp\left[i\tau w - \frac{|\tau||z|^2}{2}\right] L_m^{(n-1)}\left(|\tau||z|^2\right),$$

$\tau \neq 0,\ m \in \mathbb{N}$, or

$$\phi_\mu(z,w) = \Gamma(n) J_{n-1}(\mu|z|) \left(\frac{\mu|z|}{2}\right)^{1-n},\quad \mu \geq 0,$$

Isomorphisms:

$$\begin{aligned}\sigma(\mathrm{C}^*(H/U)) &\cong \text{the set of bounded spherical functions } \phi \\ &\cong \{(\tau, m|\tau|) : \tau \in \mathbb{R}\setminus\{0\}, m \in \mathbb{N}\} \cup \{(0,\mu) : \mu \geq 0\}\end{aligned}$$

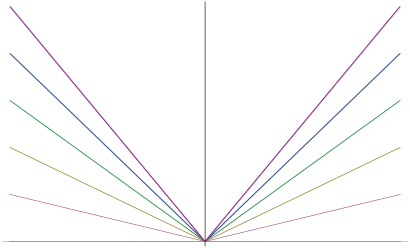

Figure : A homeomorphic image of the Gelfand spectrum $\sigma(\mathrm{C}^*(H/U))$ – the state space of *the quantum Bessel process*

Transition probabilities of Biane's quantum Bessel process

- If $\mathbf{x} = (s, k|s|)$, $s < 0$, and $u = s + t < 0$, then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{l=k}^{\infty} \frac{\Gamma(n+l)}{\Gamma(n+k)(l-k)!} \left(\frac{u}{s}\right)^{n+k} \left(1 - \frac{u}{s}\right)^{l-k} \delta_{(u,-lu)}(\mathrm{d}\mathbf{y}).$$

- If $\mathbf{x} = (s, k|s|)$ with $s < 0$ and $t = -s$, then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \frac{1}{\Gamma(n+k)} \exp\left(-\frac{y_1}{t}\right) \left(\frac{y_1}{t}\right)^{n+k-1} \frac{1}{t} \left(\delta_0 \otimes \mathrm{Leb}\right)(\mathrm{d}\mathbf{y})$$

for $\mathbf{y} = (y_0, y_1)$ from the Heisenberg fan (Leb denotes the one-dimensional Lebesgue measure).

- If $\mathbf{x} = (s, k|s|)$, $s < 0$, and $u = s + t > 0$, then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{l=0}^{\infty} \frac{\Gamma(n+k+l)}{\Gamma(n+k)l!} \left(\frac{u}{t}\right)^{n+k} \left(-\frac{s}{t}\right)^{l} \delta_{(u,lu)}(\mathrm{d}\mathbf{y}).$$

Transition probabilities of Biane's quantum Bessel process - cont.

- If $\mathbf{x} = (0, y_1)$ with $y_1 \geq 0$, then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{y_1}{t}\right)^l \exp\left(-\frac{y_1}{t}\right) \delta_{(t,lt)}(\mathrm{d}\mathbf{y}).$$

- If $\mathbf{x} = (s, k|s|)$ with $s > 0$, and $u = s + t$, then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{l=0}^{k} \binom{k}{l} \left(\frac{s}{u}\right)^l \left(1 - \frac{s}{u}\right)^{k-l} \delta_{(u,lu)}(\mathrm{d}\mathbf{y}).$$

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Multidimensional analogue of Biane's construction

.

- Take V- a simple Euclidean Jordan algebra.
- Take its complexification $W = V^{\mathbb{C}}$.
- Build a Heisenberg group $H = W \times \mathbb{R}$.
- Take $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

This will result in a Markov semigroup of transition probabilities on the Gelfand spectrum of $\mathrm{C}^*(H/U)$, which is a commutative sub-C^*-algebra of $\mathrm{C}^*(H)$. Finally, we will find an embedding of the spectrum into a subset of a Euclidean space (a *multidimensional Heisenberg fan*). Thus we will obtain a multidimensional Markov process (*the multidimensional quantum Bessel process*).

Let V be a finite dimensional commutative algebra (real or complex) with neutral element e. We say that V is Jordan algebra if

$$x^2(xy) = x(x^2y).$$

A real Jordan algebra is called Euclidean if there is a scalar product on V such that

$$(xy|z) = (y|xz) \quad \forall x, y, z \in V.$$

Basic example: the space $V = \mathrm{Sym}(m, \mathbb{R})$ of real symmetric matrices with multiplication

$$x \circ y = \frac{xy + yx}{2},$$

and scalar product $(x|y) = \mathrm{tr}(xy)$ is a simple Euclidean Jordan algebra.

Let V be a finite dimensional commutative algebra (real or complex) with neutral element e. We say that V is Jordan algebra if

$$x^2(xy) = x(x^2y).$$

A real Jordan algebra is called Euclidean if there is a scalar product on V such that

$$(xy|z) = (y|xz) \quad \forall x, y, z \in V.$$

Basic example: the space $V = \mathrm{Sym}(m, \mathbb{R})$ of real symmetric matrices with multiplication

$$x \circ y = \frac{xy + yx}{2},$$

and scalar product $(x|y) = \mathrm{tr}(xy)$ is a simple Euclidean Jordan algebra.

Jordan algebras

Let V be a finite dimensional commutative algebra (real or complex) with neutral element e. We say that V is Jordan algebra if

$$x^2(xy) = x(x^2y).$$

A real Jordan algebra is called Euclidean if there is a scalar product on V such that

$$(xy|z) = (y|xz) \quad \forall x, y, z \in V.$$

Basic example: the space $V = \mathrm{Sym}(m, \mathbb{R})$ of real symmetric matrices with multiplication

$$x \circ y = \frac{xy + yx}{2},$$

and scalar product $(x|y) = \mathrm{tr}(xy)$ is a simple Euclidean Jordan algebra.

For any Jordan algebra V we can define its rank r and trace $\mathrm{tr}(x)$ and determinant $\Delta(x)$ of $x \in V$.

If V is Euclidean Jordan algebra then for any $x \in$V there exists a Jordan frame i.e (maximal) sequence $c_1, c_2, \ldots, c_r$ of primitive orthogonal idempotents ($c_i c_j = \delta_i^j c_j$), and a unique up to permutation sequence of real numbers $\lambda_1, ..., \lambda_r$ such that

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_r c_r.$$

$$\begin{aligned} \mathrm{tr}\, x &= \lambda_1 + \lambda_2 + \cdots + \lambda_r, \\ \Delta(x) &= \lambda_1 \cdot \lambda_2 \cdots \lambda_r. \end{aligned}$$

For any Jordan algebra V we can define its rank r and trace $\mathrm{tr}(x)$ and determinant $\Delta(x)$ of $x \in V$.
If V is Euclidean Jordan algebra then for any $x \in$V there exists a Jordan frame i.e (maximal) sequence $c_1, c_2, \ldots, c_r$ of primitive orthogonal idempotents ($c_i c_j = \delta_i^j c_j$), and a unique up to permutation sequence of real numbers $\lambda_1, ..., \lambda_r$ such that

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_r c_r.$$

$$\begin{aligned} \mathrm{tr}\, x &= \lambda_1 + \lambda_2 + \cdots + \lambda_r, \\ \Delta(x) &= \lambda_1 \cdot \lambda_2 \cdots \lambda_r. \end{aligned}$$

For any Jordan algebra V we can define its rank r and trace $\mathrm{tr}(x)$ and determinant $\Delta(x)$ of $x \in V$.
If V is Euclidean Jordan algebra then for any $x \in$V there exists a Jordan frame i.e (maximal) sequence $c_1, c_2, \ldots, c_r$ of primitive orthogonal idempotents ($c_i c_j = \delta_i^j c_j$), and a unique up to permutation sequence of real numbers $\lambda_1, ..., \lambda_r$ such that

$$x = \lambda_1 c_1 + \lambda_2 c_2 + \ldots + \lambda_r c_r.$$

$$\begin{aligned} \mathrm{tr}\, x &= \lambda_1 + \lambda_2 + \cdots + \lambda_r, \\ \Delta(x) &= \lambda_1 \cdot \lambda_2 \cdots \lambda_r. \end{aligned}$$

Pierce decomposition and classification

Any Euclidean Jordan algebra has a Pierce decomposition with respect to a fixed Jordan frame,

$$V = \bigoplus_{1 \le i \le j \le m} V_{ij},$$

where $V_{ii} = \mathbb{R}c_i$ is one dimensional. If V is simple Euclidean

$$d = \dim V_{ij}, \text{ for } i \neq j.$$

does not depend on (i, j) and is called the Pierce constant of V.

V	$n = \dim(V)$	d	r
$\mathrm{Sym}(m, \mathbb{R})$	$m(m+1)/2$	1	m
$\mathrm{Herm}(m, \mathbb{C})$	m^2	2	m
$\mathrm{Herm}(m, \mathbb{H})$	$m(2m-1)$	4	m
$\mathrm{Herm}(3, \mathbb{O})$	27	8	3
$\mathbb{R} \times \mathbb{R}^{l-1}$	l	$l-2$	2

Table :

Pierce decomposition and classification

Any Euclidean Jordan algebra has a Pierce decomposition with respect to a fixed Jordan frame,

$$V = \bigoplus_{1 \le i \le j \le m} V_{ij},$$

where $V_{ii} = \mathbb{R} c_i$ is one dimensional. If V is simple Euclidean

$$d = \dim V_{ij}, \ \text{for } i \neq j.$$

does not depend on (i, j) and is called the Pierce constant of V.

V	$n = \dim(V)$	d	r
$\mathrm{Sym}(m, \mathbb{R})$	$m(m+1)/2$	1	m
$\mathrm{Herm}(m, \mathbb{C})$	m^2	2	m
$\mathrm{Herm}(m, \mathbb{H})$	$m(2m-1)$	4	m
$\mathrm{Herm}(3, \mathbb{O})$	27	8	3
$\mathbb{R} \times \mathbb{R}^{l-1}$	l	$l-2$	2

Table :

The Cone of squares

If V is Euclidean Jordan algebra then the set

$$\Omega = \mathrm{Int}\{x^2 : x \in V\},$$

is an open convex symmetric cone.
Define

$$G(\Omega) = \{g \in GL(V) : g\Omega = \Omega\}$$

$$K = G(\Omega)^0 \cap O(V)$$

Group K acts transitively on the set of Jordan frames.

If V is Euclidean Jordan algebra then the set

$$\Omega = \mathrm{Int}\{x^2 : x \in V\},$$

is an open convex symmetric cone.
Define

$$G(\Omega) = \{g \in GL(V) : g\Omega = \Omega\}$$

$$K = G(\Omega)^0 \cap O(V)$$

Group K acts transitively on the set of Jordan frames.

The Cone of squares

If V is Euclidean Jordan algebra then the set

$$\Omega = \mathrm{Int}\{x^2 : x \in V\},$$

is an open convex symmetric cone.
Define

$$G(\Omega) = \{g \in GL(V) : g\Omega = \Omega\}$$

$$K = G(\Omega)^0 \cap O(V)$$

Group K acts transitively on the set of Jordan frames.

Spherical polynomials

Let V be a simple Euclidean Jordan algebra and $W = V + iV$ its complexification. Consider the space $\mathcal{P}(W)$ of complex polynomials on W.

There is a natural representation π of $U = (\mathrm{Str}(W) \cap U(W))^0$ on $\mathcal{P}(W)$

$$(\pi(g)p)(z) = p(g^{-1}z), \; g \in U, \; p \in \mathcal{P}(W).$$

There is decomposition into invariant subspaces

$$\mathcal{P}(W) = \bigoplus_{\mathbf{m} \in \mathrm{Part(r)}} \mathcal{P}_{\mathbf{m}},$$

where $\mathrm{Part}(r) = \{(m_k)_k \in \mathbb{Z}^r : m_1 \geq m_2 \geq \ldots \geq m_r \geq 0\}$ and $(\pi, \mathcal{P}_{\mathbf{m}})$ are irreducible, pairwise inequivalent, representations of U. Let d_m denote $d_{\mathbf{m}} = \mathrm{dim}\mathcal{P}_{\mathbf{m}}$

Spherical polynomials

Let V be a simple Euclidean Jordan algebra and $W = V + iV$ its complexification. Consider the space $\mathcal{P}(W)$ of complex polynomials on W.

There is a natural representation π of $U = (\mathrm{Str}(W) \cap U(W))^0$ on $\mathcal{P}(W)$

$$(\pi(g)p)(z) = p(g^{-1}z), \ g \in U, \ p \in \mathcal{P}(W).$$

There is decomposition into invariant subspaces

$$\mathcal{P}(W) = \bigoplus_{\mathbf{m} \in \mathrm{Part(r)}} \mathcal{P}_{\mathbf{m}},$$

where $\mathrm{Part}(r) = \{(m_k)_k \in \mathbb{Z}^r : m_1 \geq m_2 \geq \ldots \geq m_r \geq 0\}$ and $(\pi, \mathcal{P}_{\mathbf{m}})$ are irreducible, pairwise inequivalent, representations of U.

Let d_m denote $d_{\mathbf{m}} = \dim \mathcal{P}_{\mathbf{m}}$

Spherical polynomials

Let V be a simple Euclidean Jordan algebra and $W = V + iV$ its complexification. Consider the space $\mathcal{P}(W)$ of complex polynomials on W.

There is a natural representation π of $U = (\mathrm{Str}(W) \cap U(W))^0$ on $\mathcal{P}(W)$

$$(\pi(g)p)(z) = p(g^{-1}z), \ g \in U, \ p \in \mathcal{P}(W).$$

There is decomposition into invariant subspaces

$$\mathcal{P}(W) = \bigoplus_{\mathbf{m} \in \mathrm{Part(r)}} \mathcal{P}_{\mathbf{m}},$$

where $\mathrm{Part}(r) = \{(m_k)_k \in \mathbb{Z}^r : m_1 \geq m_2 \geq \ldots \geq m_r \geq 0\}$ and $(\pi, \mathcal{P}_{\mathbf{m}})$ are irreducible, pairwise inequivalent, representations of U. Let d_m denote $d_{\mathbf{m}} = \dim \mathcal{P}_{\mathbf{m}}$

Spherical polynomials

Each subspace $\mathcal{P}_{\mathbf{m}}$ contains a one dimensional subspace of K-invariant polynomials. The unique K invariant polynomial $p \in \mathcal{P}_{\mathbf{m}}$ such that $p(e) = 1$ is called *spherical polynomial* and it is denoted $\Phi_{\mathbf{m}}(x)$.

Up to normalization,

- If $V = \mathrm{Sym}(m, \mathbb{R})$ then $\Phi_{\mathbf{m}} =$ zonal polynomial.
- if $V = \mathrm{Herm}(m, \mathbb{C})$, then $\Phi_{\mathbf{m}} =$ Schur polynomial,
- for any Euclidean Jordan algebra, $\Phi_{\mathbf{m}}$ are special cases of the *Jack polynomials*.

Spherical polynomials

Each subspace $\mathcal{P}_{\mathbf{m}}$ contains a one dimensional subspace of K-invariant polynomials. The unique K invariant polynomial $p \in \mathcal{P}_{\mathbf{m}}$ such that $p(e) = 1$ is called *spherical polynomial* and it is denoted $\Phi_{\mathbf{m}}(x)$.

Up to normalization,

- If $V = \mathrm{Sym}(m, \mathbb{R})$ then $\Phi_{\mathbf{m}} =$ zonal polynomial.
- if $V = \mathrm{Herm}(m, \mathbb{C})$, then $\Phi_{\mathbf{m}} =$ Schur polynomial,
- for any Euclidean Jordan algebra, $\Phi_{\mathbf{m}}$ are special cases of the *Jack polynomials*.

Spherical polynomials

Each subspace $\mathcal{P}_{\mathbf{m}}$ contains a one dimensional subspace of K-invariant polynomials. The unique K invariant polynomial $p \in \mathcal{P}_{\mathbf{m}}$ such that $p(e) = 1$ is called *spherical polynomial* and it is denoted $\Phi_{\mathbf{m}}(x)$.

Up to normalization,

- If $V = \mathrm{Sym}(m, \mathbb{R})$ then $\Phi_{\mathbf{m}} =$ zonal polynomial.
- if $V = \mathrm{Herm}(m, \mathbb{C})$, then $\Phi_{\mathbf{m}} =$ Schur polynomial,
- for any Euclidean Jordan algebra, $\Phi_{\mathbf{m}}$ are special cases of the *Jack polynomials*.

Spherical polynomials

Each subspace $\mathcal{P}_{\mathbf{m}}$ contains a one dimensional subspace of K-invariant polynomials. The unique K invariant polynomial $p \in \mathcal{P}_{\mathbf{m}}$ such that $p(e) = 1$ is called *spherical polynomial* and it is denoted $\Phi_{\mathbf{m}}(x)$.

Up to normalization,

- If $V = \mathrm{Sym}(m, \mathbb{R})$ then $\Phi_{\mathbf{m}} =$ zonal polynomial.
- if $V = \mathrm{Herm}(m, \mathbb{C})$, then $\Phi_{\mathbf{m}} =$ Schur polynomial,
- for any Euclidean Jordan algebra, $\Phi_{\mathbf{m}}$ are special cases of the *Jack polynomials*.

Spherical polynomials

Each subspace $\mathcal{P}_{\mathbf{m}}$ contains a one dimensional subspace of K-invariant polynomials. The unique K invariant polynomial $p \in \mathcal{P}_{\mathbf{m}}$ such that $p(e) = 1$ is called *spherical polynomial* and it is denoted $\Phi_{\mathbf{m}}(x)$.
Up to normalization,

- If $V = \mathrm{Sym}(m, \mathbb{R})$ then $\Phi_{\mathbf{m}} =$ zonal polynomial.
- if $V = \mathrm{Herm}(m, \mathbb{C})$, then $\Phi_{\mathbf{m}} =$ Schur polynomial,
- for any Euclidean Jordan algebra, $\Phi_{\mathbf{m}}$ are special cases of the *Jack polynomials*.

Gamma Function

Generalized Gamma function:

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} x} \Delta_{\mathbf{s}}(x) \Delta(x)^{-n/r} \mathrm{d}x, \quad \mathbf{s} \in \mathbb{C}^r$$

For $\mathrm{Re}(s_j) > (j-1)\frac{d}{2}$ the integral is convergent and is equal to

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n-r)/2} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\frac{d}{2}\right).$$

Pochhammer symbol

$$(\mathbf{s})_{\mathbf{m}} = \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})} = \prod_{i=1}^{r} \left(s_i - (i-1)\frac{d}{2}\right)_{m_i}$$

Gamma Function

Generalized Gamma function:

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} x} \Delta_{\mathbf{s}}(x) \Delta(x)^{-n/r} \mathrm{d}x, \quad \mathbf{s} \in \mathbb{C}^r$$

For $\mathrm{Re}(s_j) > (j-1)\frac{d}{2}$ the integral is convergent and is equal to

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n-r)/2} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\frac{d}{2}\right).$$

Pochhammer symbol

$$(\mathbf{s})_{\mathbf{m}} = \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})} = \prod_{i=1}^{r} \left(s_i - (i-1)\frac{d}{2}\right)_{m_i}$$

Gamma Function

Generalized Gamma function:

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega e^{-\operatorname{tr} x} \Delta_{\mathbf{s}}(x) \Delta(x)^{-n/r} \mathrm{d}x, \quad \mathbf{s} \in \mathbb{C}^r$$

For $\mathrm{Re}(s_j) > (j-1)\frac{d}{2}$ the integral is convergent and is equal to

$$\Gamma_\Omega(\mathbf{s}) = (2\pi)^{(n-r)/2} \prod_{j=1}^{r} \Gamma\left(s_j - (j-1)\frac{d}{2}\right).$$

Pochhammer symbol

$$(\mathbf{s})_{\mathbf{m}} = \frac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})} = \prod_{i=1}^{r} \left(s_i - (i-1)\frac{d}{2}\right)_{m_i}$$

Binomial coefficients and Laguerre polynomials

For any two partitions **n**, **m** of length $\leq r$ the generalized binomial coefficient is defined by equality:

$$\Phi_{\mathbf{m}}(e+x) = \sum_{\mathbf{n}} \binom{\mathbf{m}}{\mathbf{n}} \Phi_{\mathbf{n}}(x).$$

Generalized Laguerre polynomials:

$$L^{\nu}_{\mathbf{m}}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{k}| \leq |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x).$$

Binomial coefficients and Laguerre polynomials

For any two partitions $\mathbf{n}$, $\mathbf{m}$ of length $\leq r$ the generalized binomial coefficient is defined by equality:

$$\Phi_{\mathbf{m}}(e+x) = \sum_{\mathbf{n}} \binom{\mathbf{m}}{\mathbf{n}} \Phi_{\mathbf{n}}(x).$$

Generalized Laguerre polynomials:

$$L^{\nu}_{\mathbf{m}}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{k}| \leq |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{1}{(\nu)_{\mathbf{k}}} \Phi_{\mathbf{k}}(-x).$$

Multidimensional quantum Bessel process

Let $H = W \times \mathbb{R}$ be a Heisenberg group build from the complexification of a simple Euclidean Jordan algebra and $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.

Hence $(Q_t)_t$ such that $Q_t(f)(z,w) = e^{t\psi(z,w)} f(z,w)$ is a semigroup of positive contractions on commutative C^* algebra $\mathrm{C}^*(H/U)$.

V	Ω	$W = V^{\mathbb{C}}$	U
$\mathrm{Sym}(m,\mathbb{R})$	$\Pi_m(\mathbb{R})$	$\mathrm{Sym}(m,\mathbb{C})$	$U(m)$
$\mathrm{Herm}(m,\mathbb{C})$	$\Pi_m(\mathbb{C})$	$M(m,\mathbb{C})$	$U(m) \times U(m)$
$\mathrm{Herm}(m,\mathbb{H})$	$\Pi_m(\mathbb{H})$	$\mathrm{Skew}(2m,\mathbb{C})$	$U(2m)$
$\mathrm{Herm}(3,\mathbb{O})$	$\Pi_3(\mathbb{O})$	$\mathrm{Herm}(3,\mathbb{O}) \otimes \mathbb{C}$	$E_6 \times \mathbb{T}$
$\mathbb{R}^l$	Λ_l	$\mathbb{C}^l$	$SO(l) \times \mathbb{T}$

Multidimensional quantum Bessel process

Let $H = W \times \mathbb{R}$ be a Heisenberg group build from the complexification of a simple Euclidean Jordan algebra and $U = (\mathrm{Str}(W) \cap U(W))^0$, then $(U \ltimes H, U)$ is a Gelfand pair i.e. $L^1(H/U)$ is a commutative algebra.
Hence $(Q_t)_t$ such that $Q_t(f)(z,w) = e^{t\psi(z,w)} f(z,w)$ is a semigroup of positive contractions on commutative C^* algebra $\mathrm{C}^*(H/U)$.

V	Ω	$W = V^{\mathbb{C}}$	U
$\mathrm{Sym}(m,\mathbb{R})$	$\Pi_m(\mathbb{R})$	$\mathrm{Sym}(m,\mathbb{C})$	$U(m)$
$\mathrm{Herm}(m,\mathbb{C})$	$\Pi_m(\mathbb{C})$	$M(m,\mathbb{C})$	$U(m) \times U(m)$
$\mathrm{Herm}(m,\mathbb{H})$	$\Pi_m(\mathbb{H})$	$\mathrm{Skew}(2m,\mathbb{C})$	$U(2m)$
$\mathrm{Herm}(3,\mathbb{O})$	$\Pi_3(\mathbb{O})$	$\mathrm{Herm}(3,\mathbb{O}) \otimes \mathbb{C}$	$E_6 \times \mathbb{T}$
$\mathbb{R}^l$	Λ_l	$\mathbb{C}^l$	$SO(l) \times \mathbb{T}$

Bounded spherical functions:

Spectrum of $C^*(H/U)$ can be identified with the set $\Sigma = \Sigma_1 \cup \Sigma_2$ of bounded U spherical function where

$$\Sigma_1 = \{\varphi(\mu, \mathbf{m}; \cdot, \cdot) : \mu \in \mathbb{R} \setminus \{0\},\ \mathbf{m} \text{ - a partition of length} \leq r\},$$

and

$$\Sigma_2 = \{\varphi(\tau; \cdot, \cdot) : \tau = (\tau_1, \ldots, \tau_r) \in \mathbb{R}^r, \tau_1 \geq \ldots \geq \tau_r \geq 0\}.$$

$$\varphi(\mu, \mathbf{m}; z, w) = \frac{1}{(n/r)_{\mathbf{m}}} \exp\left(i\mu w - \frac{|\mu|\|z\|^2}{2}\right) L_{\mathbf{m}}^{n/r}(|\mu| v^2),$$

where $v \in V$ is such that $z = u(v)$ for $u \in U$.

$$\varphi(\tau; z, w) = \sum_{\mathbf{k} \geq 0} \frac{d_{\mathbf{k}}}{((n/r)_{\mathbf{k}})^2} (-1)^{|\mathbf{k}|} \Phi_{\mathbf{k}}(\tau) \Phi_{\mathbf{k}}\left(v^2\right);$$

Bounded spherical functions:

Spectrum of $C^*(H/U)$ can be identified with the set $\Sigma = \Sigma_1 \cup \Sigma_2$ of bounded U spherical function where

$$\Sigma_1 = \{\varphi(\mu, \mathbf{m}; \cdot, \cdot) : \mu \in \mathbb{R} \setminus \{0\},\ \mathbf{m} \text{ - a partition of length} \leq r\},$$

and

$$\Sigma_2 = \{\varphi(\tau; \cdot, \cdot) : \tau = (\tau_1, \dots, \tau_r) \in \mathbb{R}^r, \tau_1 \geq \dots \geq \tau_r \geq 0\}.$$

$$\varphi(\mu, \mathbf{m}; z, w) = \frac{1}{(n/r)_{\mathbf{m}}} \exp\left(i\mu w - \frac{|\mu| \|z\|^2}{2}\right) L_{\mathbf{m}}^{n/r}(|\mu| v^2),$$

where $v \in V$ is such that $z = u(v)$ for $u \in U$.

$$\varphi(\tau; z, w) = \sum_{\mathbf{k} \geq 0} \frac{d_{\mathbf{k}}}{((n/r)_{\mathbf{k}})^2} (-1)^{|\mathbf{k}|} \Phi_{\mathbf{k}}(\tau) \Phi_{\mathbf{k}}\left(v^2\right);$$

One can show that the transformation of Σ into $\mathbb{R}^{r+1}$ via

$$\Sigma_1 \ni \varphi(\mu, \mathbf{m}; \cdot, \cdot) \mapsto (\mu, \mu m_1, \ldots, \mu m_r),$$

$$\Sigma_2 \ni \varphi(\tau; \cdot, \cdot) \mapsto (0, \tau_1, \ldots, \tau_r)$$

is a homeomorphism onto its image.
The image $\Sigma' = \Sigma'(r)$ of Σ under this map, given explicitly as

$$\begin{aligned} \Sigma &= \{(s, |s|k_1, \ldots, |s|k_r) : s < 0, \kappa = (k_1, \ldots, k_r) \in \mathrm{Part}(r)\} \\ &\cup \{(0, \tau_1, \ldots, \tau_r) : \tau_1 \geq \tau_2 \geq \ldots \geq \tau_r \geq 0\} \\ &\cup \{(s, |s|k_1, \ldots, |s|k_r) : s > 0, \kappa \in \mathrm{Part}(r)\}. \end{aligned}$$

(*the multidimensional Heisenberg fan*)

Transition probabilities for multidimensional Quantum Bessel Process

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$, $t \geq 0$ and $\mu = \kappa + t < 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{|\mathbf{m}| \geq |\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{d_{\mathbf{m}}}{d_{\mathbf{k}}} \left(1 - \frac{\mu}{\kappa}\right)^{|\mathbf{m}| - |\mathbf{k}|} \left(\frac{\mu}{\kappa}\right)^{|\mathbf{k}| + n} \delta_{(\mu, -\mu\mathbf{m})}(\mathrm{d}\mathbf{y}).$$

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$ and $t = -\kappa$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \frac{t^{-n} r! \left(\Gamma(1 + d/2)\right)^r}{(n/r)_{\mathbf{k}} \prod_{j=1}^{r} \Gamma\left(1 + (j-1)d/2\right)\Gamma(1 + jd/2)} \exp\left(-\operatorname{tr}(a/t)\right)$$

$$\times \Phi_{\mathbf{k}}(a/t) \prod_{1 \leq i < j \leq r} (a_j - a_i)^d \mathbb{1}_{R_{++}}(a) \left(\delta_0 \otimes \lambda^{\otimes r}\right)(\mathrm{d}\mathbf{y})$$

where a is an element of

$$R_{++} = \{a \in \mathbb{R}^r : a_1 > \ldots > a_r > 0\},$$

$\mathbf{y} = (a_0, a_1, \ldots, a_r)$, and λ denotes the one-dimensional Lebesgue measure.

Transition probabilities for multidimensional Quantum Bessel Process

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$, $t \geq 0$ and $\mu = \kappa + t < 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{|\mathbf{m}| \geq |\mathbf{k}|} \binom{\mathbf{m}}{\mathbf{k}} \frac{d_{\mathbf{m}}}{d_{\mathbf{k}}} \left(1 - \frac{\mu}{\kappa}\right)^{|\mathbf{m}|-|\mathbf{k}|} \left(\frac{\mu}{\kappa}\right)^{|\mathbf{k}|+n} \delta_{(\mu, -\mu\mathbf{m})}(\mathrm{d}\mathbf{y}).$$

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$ and $t = -\kappa$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \frac{t^{-n} r! \left(\Gamma(1 + d/2)\right)^r}{(n/r)_{\mathbf{k}} \prod_{j=1}^{r} \Gamma\left(1 + (j-1)d/2)\right)\Gamma(1 + jd/2)} \exp\left(-\operatorname{tr}(a/t)\right)$$
$$\times \Phi_{\mathbf{k}}(a/t) \prod_{1 \leq i < j \leq r} (a_j - a_i)^d \mathbb{1}_{R_{++}}(a) \left(\delta_0 \otimes \lambda^{\otimes r}\right)(\mathrm{d}\mathbf{y})$$

where a is an element of

$$R_{++} = \{a \in \mathbb{R}^r : a_1 > \ldots > a_r > 0\},$$

$\mathbf{y} = (a_0, a_1, \ldots, a_r)$, and λ denotes the one-dimensional Lebesgue measure.

Multidimensional Quantum Bessel Process - cont'd

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$ and $\mu = \kappa + t > 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \left(\frac{\mu}{t}\right)^{|\mathbf{k}|+n} \left(1 - \frac{\mu}{t}\right)^{|\mathbf{m}|} \sum \frac{\binom{\mathbf{k}}{\mathbf{n}}\binom{\mathbf{m}}{\mathbf{n}}}{d_{\mathbf{n}}} \delta_{(\mu,\mu\mathbf{m})}(\mathrm{d}\mathbf{y}),$$

where the last summation is over partitions $\mathbf{n}$ of the consecutive integers ranging from 0 to $\min(|\mathbf{k}|, |\mathbf{m}|)$. If $\mathbf{x} = (0, \tau_1, \ldots, \tau_r)$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{\mathbf{m}\geq 0} \frac{d_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \Phi_{\mathbf{m}}(\tau/t) e^{-\operatorname{tr}(\tau/t)} \delta_{(t,t\mathbf{m})}(\mathrm{d}\mathbf{y})$$

If $\mathbf{x} = (\kappa, \kappa\mathbf{k})$ is such that $\kappa > 0$, $t \geq 0$ and $\mu = \kappa + t > 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{|\mathbf{m}|\leq|\mathbf{k}|} \binom{\mathbf{k}}{\mathbf{m}} \left(\frac{\kappa}{\mu}\right)^{|\mathbf{m}|} \left(1 - \frac{\kappa}{\mu}\right)^{|\mathbf{k}|-|\mathbf{m}|} \delta_{(\mu,\mu\mathbf{m})}(\mathrm{d}\mathbf{y}).$$

Multidimensional Quantum Bessel Process - cont'd

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$ and $\mu = \kappa + t > 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \left(\frac{\mu}{t}\right)^{|\mathbf{k}|+n} \left(1 - \frac{\mu}{t}\right)^{|\mathbf{m}|} \sum \frac{\binom{\mathbf{k}}{\mathbf{n}}\binom{\mathbf{m}}{\mathbf{n}}}{d_{\mathbf{n}}} \delta_{(\mu,\mu\mathbf{m})}(\mathrm{d}\mathbf{y}),$$

where the last summation is over partitions $\mathbf{n}$ of the consecutive integers ranging from 0 to $\min(|\mathbf{k}|, |\mathbf{m}|)$. If $\mathbf{x} = (0, \tau_1, \ldots, \tau_r)$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{\mathbf{m}\geq 0} \frac{d_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \Phi_{\mathbf{m}}(\tau/t) e^{-\operatorname{tr}(\tau/t)} \delta_{(t,t\mathbf{m})}(\mathrm{d}\mathbf{y})$$

If $\mathbf{x} = (\kappa, \kappa\mathbf{k})$ is such that $\kappa > 0$, $t \geq 0$ and $\mu = \kappa + t > 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{|\mathbf{m}|\leq|\mathbf{k}|} \binom{\mathbf{k}}{\mathbf{m}} \left(\frac{\kappa}{\mu}\right)^{|\mathbf{m}|} \left(1 - \frac{\kappa}{\mu}\right)^{|\mathbf{k}|-|\mathbf{m}|} \delta_{(\mu,\mu\mathbf{m})}(\mathrm{d}\mathbf{y}).$$

Multidimensional Quantum Bessel Process - cont'd

If $\mathbf{x} = (\kappa, -\kappa\mathbf{k})$ with $\kappa < 0$ and $\mu = \kappa + t > 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \left(\frac{\mu}{t}\right)^{|\mathbf{k}|+n} \left(1 - \frac{\mu}{t}\right)^{|\mathbf{m}|} \sum \frac{\binom{\mathbf{k}}{\mathbf{n}}\binom{\mathbf{m}}{\mathbf{n}}}{d_{\mathbf{n}}} \delta_{(\mu,\mu\mathbf{m})}(\mathrm{d}\mathbf{y}),$$

where the last summation is over partitions $\mathbf{n}$ of the consecutive integers ranging from 0 to $\min(|\mathbf{k}|, |\mathbf{m}|)$. If $\mathbf{x} = (0, \tau_1, \ldots, \tau_r)$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{\mathbf{m}\geq 0} \frac{d_{\mathbf{m}}}{(n/r)_{\mathbf{m}}} \Phi_{\mathbf{m}}(\tau/t) e^{-\operatorname{tr}(\tau/t)} \delta_{(t,t\mathbf{m})}(\mathrm{d}\mathbf{y})$$

If $\mathbf{x} = (\kappa, \kappa\mathbf{k})$ is such that $\kappa > 0$, $t \geq 0$ and $\mu = \kappa + t > 0$ then

$$q_t(\mathbf{x}, \mathrm{d}\mathbf{y}) = \sum_{|\mathbf{m}|\leq|\mathbf{k}|} \binom{\mathbf{k}}{\mathbf{m}} \left(\frac{\kappa}{\mu}\right)^{|\mathbf{m}|} \left(1 - \frac{\kappa}{\mu}\right)^{|\mathbf{k}|-|\mathbf{m}|} \delta_{(\mu,\mu\mathbf{m})}(\mathrm{d}\mathbf{y}).$$

A process on partitions

Let $\mathbf{X}_t = (t, X_t)_t$ be a multidimensional Quantum Bessel Process.
Define

$$Y_t = \frac{X_t}{|t|}$$

Then a time transformation of the process Y_t gives a homogenous Markov process R_t on the set of partitions $\mathrm{Part}(r)$.
Its semigroup can be constructed from a semigroup of positive contraction on commutative sub-C^*-algebra of $\mathcal{K}(L_2(\mathbb{R}^{2n}))$

A process on partitions

Let $\mathbf{X}_t = (t, X_t)_t$ be a multidimensional Quantum Bessel Process. Define

$$Y_t = \frac{X_t}{|t|}$$

Then a time transformation of the process Y_t gives a homogenous Markov process R_t on the set of partitions $\mathrm{Part}(r)$.

Its semigroup can be constructed from a semigroup of positive contraction on commutative sub-C^*-algebra of $\mathcal{K}(L_2(\mathbb{R}^{2n}))$

A process on partitions

Let $\mathbf{X}_t = (t, X_t)_t$ be a multidimensional Quantum Bessel Process. Define

$$Y_t = \frac{X_t}{|t|}$$

Then a time transformation of the process Y_t gives a homogenous Markov process R_t on the set of partitions $\mathrm{Part}(r)$.
Its semigroup can be constructed from a semigroup of positive contraction on commutative sub-C^*-algebra of $\mathcal{K}(L_2(\mathbb{R}^{2n}))$

One can show that $|R_t|$ is a pure birth process (Yule process). We can think of R_t as a pure birth process which describes a population of $|m|$ particles divided into r subpopulations. The largest one has m_1 particles, the second largest m_2, etc... The dynamics of the population changes is encoded in the generalized binomial coefficients associated with the simple Euclidean Jordan algebras. The birth (and death) processes originating in this way are currently under study.

Interpretations

One can show that $|R_t|$ is a pure birth process (Yule process). We can think of R_t as a pure birth process which describes a population of $|m|$ particles divided into r subpopulations. The largest one has m_1 particles, the second largest m_2, etc...
The dynamics of the population changes is encoded in the generalized binomial coefficients associated with the simple Euclidean Jordan algebras. The birth (and death) processes originating in this way are currently under study.

The end

Thank you for your attention!