General Edgeworth expansions with applications to profiles of random trees

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Assumptions on profiles

A random profile is an arbitrary sequence

$$\mathbb{L}_n = (\mathbb{L}_n(k))_{k\in\mathbb{Z}}, \quad n \in \mathbb{N}$$

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Assumption A1: There exists an open interval $(\beta_-, \beta_+) \subset \mathbb{R}$ containing zero such that for every $n \in \mathbb{N}$ and every $\beta \in (\beta_-, \beta_+)$,

$$\sum_{k\in\mathbb{Z}} |\mathbb{L}_n(k)| e^{\beta k} < \infty \quad \text{a.s.}$$

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The interval (β_-, β_+) may be infinite. For example, if for every $n \in \mathbb{N}$ the support of a profile $\{k \in \mathbb{Z} \colon \mathbb{L}_n(k) \neq 0\}$ is finite with probability one, then Assumption A1 holds with $(\beta_-, \beta_+) = \mathbb{R}$.

Assumptions on profiles

Assume that there exist

- a sequence $(w_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{n\to\infty} w_n = +\infty$;
- an open domain $\mathcal{D} \subset \{\beta \in \mathbb{C} \colon \beta_- < \operatorname{Re}\beta < \beta_+\}$ such that $\mathcal{D} \cap \mathbb{R} = (\beta_-, \beta_+)$;
- a deterministic analytic function $\varphi : \mathcal{D} \to \mathbb{C}$ such that for every $\beta \in (\beta_-, \beta_+)$ we have $\varphi(\beta) \in \mathbb{R}$ and $\varphi''(\beta) > 0$.

From Assumption A1 it follows that with probability one, the normalized Laplace transform

$$W_n(\beta) := e^{-\varphi(\beta) w_n} \sum_{k\in\mathbb{Z}} \mathbb{L}_n(k) e^{\beta k}, \quad \beta \in \mathcal{D},$$

is a random analytic function on $\mathcal{D}$ for every $n \in \mathbb{N}$.

Assumptions on profiles

Assumption A2: With probability one the sequence of random analytic functions $(W_n)_{n\in\mathbb{N}}$ converges locally uniformly on $\mathcal{D}$, as $n\to\infty$, to a random analytic function W_∞ such that $\mathbb{P}[W(\beta)\neq 0$ for all $\beta\in(\beta_-,\beta_+)]=1$.

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Assume also that the speed of convergence is superpolynomial in w_n:

Assumption A3: For every compact set $K \subset \mathcal{D}$ and every $r \in \mathbb{N}$ there exists an a.s. finite $C_{K,r}$ such that for all $n \in \mathbb{N}$,

$$\sup_{\beta \in K} |W_n(\beta) - W_\infty(\beta)| < C_{K,r} w_n^{-r}.$$

Assumption on profiles

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Assumption A4: For every compact set $K \subset (\beta_-, \beta_+)$, every $a > 0$ and $r \in \mathbb{N}_0$, we have

$$\sup_{\beta \in K} \left[e^{-\varphi(\beta) w_n} \int_a^{\pi} \left| \sum_{k \in \mathbb{Z}} \mathbb{L}_n(k) e^{k(\beta + iu)} \right| \mathrm{d}u \right] = o(w_n^{-r}) \quad \text{a.s.}$$

as $n \to \infty$.

Bell polynomials

The complete Bell polynomials $B_j(z_1, \ldots, z_j)$ are defined by the formal identity

$$\exp\left\{\sum_{j=1}^{\infty} \frac{x^j}{j!} z_j\right\} = \sum_{j=0}^{\infty} \frac{x^j}{j!} B_j(z_1, \ldots, z_j).$$

Therefore, $B_0 = 1$ and for $j \in \mathbb{N}$,

$$B_j(z_1, \ldots, z_j) = \sum{}' \frac{j!}{i_1! \ldots i_j!} \left(\frac{z_1}{1!}\right)^{i_1} \ldots \left(\frac{z_j}{j!}\right)^{i_j},$$

where the sum $\sum{}'$ is taken over all $i_1, \ldots, i_j \in \mathbb{N}_0$ such that $1i_1 + 2i_2 + \ldots + ji_j = j$.

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The first few Bell polynomials:

$$B_1(z_1) = z_1, \quad B_2(z_1, z_2) = z_1^2 + z_2 \quad B_3(z_1, z_2, z_3) = z_1^3 + 3z_1 z_2 + z_3.$$

Hermite polynomials

The "probabilist" Hermite polynomial $\mathrm{He}_n(x)$ of degree n is defined by

$$\mathrm{He}_n(x) = e^{\frac{1}{2}x^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{-\frac{1}{2}x^2}.$$

The first few Hermite polynomials are:

$$\mathrm{He}_1(x) = x, \quad \mathrm{He}_2(x) = x^2 - 1, \quad \mathrm{He}_3(x) = x^3 - 3x,$$
$$\mathrm{He}_4(x) = x^4 - 6x^2 + 3, \quad \mathrm{He}_6(x) = x^6 - 15x^4 + 45x^2 - 15.$$

The main theorem

Consider a random profile $\mathbb{L}_1, \mathbb{L}_2, \ldots$ which satisfies assumptions A1–A4.

Consider the tilted profile $k \mapsto e^{\beta k - \varphi(\beta) w_n} \mathbb{L}_n(k)$ and define its "mean" $\mu(\beta)$ and the "standard deviation" $\sigma(\beta)$:

$$\mu(\beta) = \varphi'(\beta), \quad \sigma^2(\beta) = \varphi''(\beta),$$

and "the normalized coordinates"

$$x_n(k) = x_n(k; \beta) = \frac{k - \mu(\beta) w_n}{\sigma(\beta)\sqrt{w_n}}, \quad k \in \mathbb{Z}.$$

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$$x_n(k) = x_n(k;\beta) = \frac{k - \mu(\beta) w_n}{\sigma(\beta)\sqrt{w_n}}, \quad k \in \mathbb{Z}.$$

Define the "deterministic cumulants" $\kappa_j(\beta)$ and the "random cumulants" $\chi_j(\beta)$ as follows:

$$\kappa_j(\beta) = \varphi^{(j)}(\beta), \quad \chi_j(\beta) = (\log W_\infty)^{(j)}(\beta).$$

General Edgeworth expansion

Theorem

Let $\mathbb{L}_1, \mathbb{L}_2, \dots$ *be a random profile satisfying assumptions A1–A4. Fix* $r \in \mathbb{N}_0$ *and a compact set* $K \subset (\beta_-, \beta_+)$*. Then*

$$w_n^{\frac{r+1}{2}} \sup_{k \in \mathbb{Z}} \sup_{\beta \in K} \Big| e^{\beta k - \varphi(\beta) w_n} \mathbb{L}_n(k) - \frac{W_\infty(\beta) e^{-\frac{1}{2} x_n^2(k)}}{\sigma(\beta)\sqrt{2\pi w_n}} \sum_{j=0}^{r} \frac{G_j(x_n(k);\beta)}{w_n^{j/2}} \Big| \overset{a.s.}{\to} 0,$$

where $G_j(x) = G_j(x;\beta)$, $j \in \mathbb{N}_0$ *is a polynomial of degree* $3j$, *defined by*

$$G_j(x) = \frac{(-1)^j}{j!} e^{\frac{1}{2}x^2} B_j(D_1, \dots, D_j) e^{-\frac{1}{2}x^2}.$$

Here B_j *is* j*-th Bell polynomial and* $D_1, D_2, \dots$ *are linear differential operators with random coefficients:*

$$D_j = \frac{\varphi^{(j+2)}(\beta)}{(j+1)(j+2)} \left(\frac{1}{\sigma(\beta)} \frac{\mathrm{d}}{\mathrm{d}x}\right)^{j+2} + \chi_j(\beta) \left(\frac{1}{\sigma(\beta)} \frac{\mathrm{d}}{\mathrm{d}x}\right)^{j}.$$

General Edgeworth expansion

The first three terms G_0, G_1, G_2:

$$G_0(x) = 1,$$

$$G_1(x) = \frac{\chi_1(\beta)}{\sigma(\beta)} x + \frac{\kappa_3(\beta)}{6\sigma^3(\beta)} \mathrm{He}_3(x),$$

$$G_2(x) = \frac{\chi_1^2(\beta) + \chi_2(\beta)}{2\sigma^2(\beta)} \mathrm{He}_2(x) + \left(\frac{\kappa_4(\beta)}{24\sigma^4(\beta)} + \frac{\kappa_3(\beta)\chi_1(\beta)}{6\sigma^4(\beta)} \right) \mathrm{He}_4(x) + \frac{\kappa_3^2(\beta)}{72\sigma^6(\beta)} \mathrm{He}_6(x),$$

The classical Edgeworth expansion

Let $Z_1, Z_2, \ldots$ be an i.i.d. sequence of random variables with values in $\mathbb{Z}$, mean $\mu := \mathbb{E}Z_1$, variance $\sigma^2 := \mathbb{D}Z_1 \neq 0$ and cumulant

$$\varphi(\beta) := \log \mathbb{E}e^{\beta Z_1},$$

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Consider a deterministic profile $\mathbb{L}_n$:

$$\mathbb{L}_n(k) = \mathbb{P}[Z_1 + \ldots + Z_n = k], \quad k \in \mathbb{Z}.$$

Assumptions A1–A3 hold with φ defined above, $w_n = n$ and $W_\infty(\beta) = W_n(\beta) = 1$. In particular, all cumulants χ_k vanish. Assumption A4 holds if the distribution of Z_1 is 1-arithmetic.

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From the main theorem with $\beta = 0$ we obtain the classical Chebyshev-Edgeworth-Cramer expansion:

$$\lim_{n\to\infty} n^{\frac{r+1}{2}} \sup_{k\in\mathbb{N}} \left| \mathbb{P}[Z_1 + \ldots + Z_n = k] - \frac{e^{-\frac{1}{2}x_n^2(k)}}{\sigma\sqrt{2\pi n}} \sum_{j=0}^{r} \frac{q_j(x_n(k))}{n^{j/2}} \right| = 0,$$

where q_j is a polynomial of degree $3j$ with coefficients expressible in terms of the cumulants

$\kappa_2, \ldots, \kappa_{j+2}$.

Branching random walks

Consider a system of particles on $\mathbb{Z}$ with the following evolution:

- at time 0 there is a single particle at 0;
- at each step *one* particle is chosen uniformly at random among existing particles;
- the chosen particle is replaced by a cluster of particles whose displacements w.r.t. the mother are described by a point process $\zeta = \sum_{i=1}^{N} \delta_{Z_i}$ (where N, the number of particles, is a.s. finite) on $\mathbb{Z}$;
- all random mechanisms involved in this definition are independent.

Branching random walks

Let S_n be the number of particles after n splitting events, and let their positions be $x_{1,n}, \dots, x_{S_n,n}$.

Let us denote by $\mathbb{L}_n(k)$ the number of particles at site $k \in \mathbb{Z}$ after n splitting events:

$$\mathbb{L}_n(k) = \#\{1 \le j \le S_n : x_{j,n} = k\}. \tag{1}$$

The function $k \mapsto \mathbb{L}_n(k)$ is called the *profile* of the one-split BRW.

Profiles of the one-split BRW

Denote by ν_k the expected number of particles at site $k \in \mathbb{Z}$ in the cluster process ζ:

$$\nu_k = \mathbb{E}\zeta(\{k\}) = \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}_{\{Z_i=k\}}\right], \quad k \in \mathbb{Z}.$$

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Assumption B2: The cluster point process ζ is a.s. non-empty, and the probability that it has at least 2 particles is positive. In other words, $N \geq 1$ a.s. and $\mathbb{P}[N = 1] \neq 1$.

Profiles of the one-split BRW

Denote by $m(\beta)$ the moment generating function of the intensity of the cluster point process ζ minus 1:

$$m(\beta) = \sum_{k\in\mathbb{Z}} e^{\beta k}\nu_k - 1 = \mathbb{E}\left[\sum_{i=1}^{N} e^{\beta Z_i}\right] - 1.$$

The expected number of particles at time n is $\mathbb{E}S_n = 1 + m(0)n$, where, by Assumption B2,

$$m(0) = \sum_{k\in\mathbb{Z}} \nu_k - 1 = \mathbb{E}N - 1 > 0.$$

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Assumption B4: ν is not concentrated on any proper additive subgroup of $\mathbb{Z}$. In other words, $\nu(\mathbb{Z}\backslash a\mathbb{Z}) \neq 0$ for all $a \in \{2, 3, \ldots\}$.

Profiles of the one-split BRW

Define the function

$$\varphi(\beta) = \frac{m(\beta)}{m(0)}, \quad \operatorname{Re}\beta \in \mathcal{I}$$

and denote by $(\beta_-, \beta_+) \subset \mathcal{I}$ the open interval on which $\varphi'(\beta)\beta < \varphi(\beta)$:

$$\beta_- = \inf\{\beta \in \mathcal{I} \colon \varphi'(\beta)\beta < \varphi(\beta)\}, \tag{2}$$

$$\beta_+ = \sup\{\beta \in \mathcal{I} \colon \varphi'(\beta)\beta < \varphi(\beta)\}. \tag{3}$$

The interval (β_-, β_+) is non-empty because it contains 0.

Assumption B5: For any $\beta \in (\beta_-, \beta_+)$ there is $\gamma = \gamma(\beta) > 1$ such that

$$\mathbb{E}\left[\left(\sum_{i=1}^{N} e^{\beta Z_i}\right)^{\gamma}\right] < \infty.$$

Profiles of the one-split BRW

The next theorem states that the sequence of the one-split BRW profiles satisfies Assumptions A2 and A3 with $w_n = \log n$. The proof uses an embedding into a continuous-time BRW in conjunction with well-known Biggins' results.

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Theorem

Under Assumptions B1–B3 and B5, there is an open neighborhood $\mathcal{D}$ of the interval (β_-, β_+) in the complex plane such that, with probability 1, W_n converges to some random analytic function W_∞ locally uniformly on $\mathcal{D}$. Moreover, for every compact set $K \subset \mathcal{D}$ and $r \in \mathbb{N}$ we can find an a.s. finite random variable $C_{K,r}$ such that for all $n \in \mathbb{N}$,

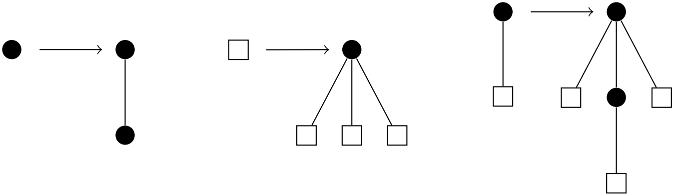
$$\sup_{\beta \in K} |W_n(\beta) - W_\infty(\beta)| < C_{K,r} (\log n)^{-r}.$$

With probability 1, the function W_∞ has no zeros on the interval (β_-, β_+).

Random trees and one-split BRW

There are nodes of two types: the external ones (denoted by □) and the internal ones (denoted by •).

Construction rules for random trees:



Left: RRT. Middle: D-ary recursive with $D = 3$.
Right: PORT.

Random trees and one-split BRW

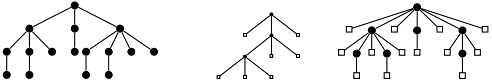


Figure: Sample realizations of random trees. Left: RRT. Middle: D-ary recursive tree with $D = 3$. Right: PORT.

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B. Chauvin, L. Devroye, M. Drmota, M. Fuchs, H.-K. Hwang, H. Mahmoud, R. Neininger, A. Panholzer etc.

Random trees and the one-split BRW

(i) Binary search tree correspond to the one-split BRW with the deterministic displacement point process $\zeta = 2\delta_1$ because at any step of the construction an external node at depth k is replaced by two new external nodes at depth $k+1$. We have

$$\varphi(\beta) = 2e^{\beta} - 1, \;\; m(0) = 1, \;\; \mu(0) = \sigma^2(0) = \kappa_j(0) = 2, \;\; j \in \mathbb{N}.$$

The constants $\beta_- \approx -1.678$ and $\beta_+ \approx 0.768$ are the solutions of $2e^{\beta}(1-\beta) = 1$.

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(ii) Random recursive trees (RRTs) correspond to the one-split BRW with the deterministic displacement point process $\zeta = \delta_0 + \delta_1$. In particular,

$$\varphi(\beta) = e^{\beta}, \ \ m(0) = 1, \ \ \mu(0) = \sigma^2(0) = \kappa_j(0) = 1, \ \ j \in \mathbb{N}.$$

We have $\beta_- = -\infty$ and $\beta_+ = 1$.

Profiles BST

Theorem

Let $(\mathbb{L}_n(k))_{k\in\mathbb{Z}}$ *be the profile of a binary search tree with* $n+1$ *external nodes. For every* $r \in \mathbb{N}_0$, *we have*

$$(\log n)^{\frac{r+1}{2}} \sup_{k\in\mathbb{Z}} \Bigg| \frac{1}{n}\mathbb{L}_n(k) - \frac{e^{-\frac{(k-2\log n)^2}{4\log n}}}{\sqrt{4\pi \log n}} \sum_{j=0}^{r} G_j\left(\frac{k-2\log n}{\sqrt{2\log n}};0\right)\frac{1}{(\log n)^{j/2}}\Bigg| \overset{a.s.}{\to} 0,$$

where $G_j(x;0)$ *is a polynomial in* x *of degree* $3j$ *whose coefficients can be linearly expressed through the derivatives* $W'_\infty(0),\dots,W^{(j)}_\infty(0)$.

Profiles BST

The *mode* u_n and the *width* M_n of a profile are defined by

$$u_n = \arg\max_{k \in \mathbb{Z}} \mathbb{L}_n(k), \quad M_n = \max_{k \in \mathbb{Z}} \mathbb{L}_n(k).$$

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Theorem

There is an a.s. finite random variable K such that for $n > K$, the mode u_n of the BST with $n+1$ external nodes is equal to one of the numbers

$$\lfloor 2 \log n + \chi_1(0) - 1/2 \rfloor \text{ or } \lceil 2 \log n + \chi_1(0) - 1/2 \rceil,$$

where $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$ denote the floor and the ceiling functions, respectively, and $\chi_1(0) = W'_\infty(0)/W_\infty(0)$.

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Actually, we can say more about the behavior of the mode. The following statements hold with probability 1:

- there are arbitrarily long intervals of consecutive n's for which the mode u_n is unique and $u_n = \lceil 2\log n + \chi_1(0) - 1/2 \rceil$;
- similarly, there are arbitrarily long intervals of consecutive n's for which u_n is unique and $u_n = \lfloor 2\log n + \chi_1(0) - 1/2 \rfloor$;
- the set of $n \in \mathbb{N}$ such that u_n is the integer closest to $2\log n + \chi_1(0) - 1/2$ has asymptotic density one.

Profiles BST

Theorem

Let M_n be the width of a binary search tree with $n+1$ external nodes. With probability 1, the set of subsequential limits of the sequence

$$\tilde{M}_n := 4\log n\left(1 - \frac{\sqrt{4\pi \log n}\, M_n}{n}\right), \quad n \in \mathbb{N},$$

is the interval

$$[\chi_2(0) - 1/12, \chi_2(0) + 1/6].$$

Set $\theta_n = \min_{k\in\mathbb{Z}} |2\log n + \chi_1(0) - 1/2 - k|$. *Then*

$$\tilde{M}_n - \theta_n^2 \overset{a.s.}{\to} \chi_2(0) - \frac{1}{12}.$$

Thank you for attention!

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